The growth of Taylor vortices in flow between rotating cylinders

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In flow between concentric rotating circular cylinders, it was shown by Taylor (1923) that instability may occur in the form of toroidal vortices spaced regularly along the axis. When the vortex motion occurs additional torque is required to keep the cylinders in motion at given speeds. Stuart (1958) used an energy-balance method, in the case when the annular gap is small compared with the radius, to estimate the additional torque and the associated finite amplitude attained by the vortices. He included the effect of distortion of the mean motion, but ignored the generation of harmonics of the fundamental mode and the distortion of the velocity associated with the fundamental mode. It is now known that these are not valid mathematical approximations and a rigorous perturbation expansion is developed here to remedy the deficiency. The analysis is valid for any gap width and any angular speeds of the containing cylinders, but requires the amplification rate of the disturbance to be small.

Numerical results using a digital computer are obtained for the shape and amplitude of the vortices in three cases: (i) when the outer cylinder has twice the radius of the inner one and is kept at rest, (ii) when the gap is small and the cylinders rotate with nearly the same speeds, and (iii) when the gap is small and the outer cylinder is kept at rest. The equilibrium amplitude obtained in the last case is substantially the same as that found by Stuart.

The results for cases (i) and (iii) give close agreement with the experimental values obtained by Taylor (1936) and Donnelly (1958) for the torque required to keep the inner cylinder rotating with constant speed while the outer one is at rest, for a certain range of speeds. In the small-gap problem it is shown that the equilibrium amplitude is almost proportional to $1-m$, where m is the ratio of the angular speeds of the outer and inner cylinders.

1. Introduction

Taylor has shown, both theoretically (1923) and experimentally (1923, 1936), that the flow between two concentric rotating circular cylinders becomes unstable if the speed of the inner cylinder is increased above a critical value. The disturbance takes the form of cellular toroidal vortices spaced regularly along the axis of the cylinders. The finite strength of the vortices is a function of the angular speeds of the cylinders, and in Taylor's (1936) experiments this was illustrated by the dependence of torque on angular speed. Taylor's theoretical analysis consisted of a mathematical examination of the conditions under which the amplitudes of cellular disturbances (of the kind observed in experiments) grow or decay with time. To this end the equations of motion are linearized for small amplitudes of disturbance, and the condition for neutral (or marginal) stability can be found. For given ratios of radii and angular speeds, the condition takes the form of a relationship between a velocity parameter (the Taylor number T) and the wave-number of the periodic disturbance. For a given wave-number the disturbance is amplified if the Taylor number lies above its critical (neutral-stability) value for that wave-number, and is damped if the Taylor number lies below that value. We may refer to the two regions as supercritical and subcritical respectively. According to linearized theory the amplification or damping of non-neutral disturbances takes place exponentially with time. The theoretical conditions for neutral stability were amply confirmed by Taylor's (1923) experimental observations, for several ratios of both the angular speeds and the cylinder radii.

An additional problem arises, however, in the (supercritical) region where linearized theory predicts a disturbance which increases exponentially with time. According to Taylor's (1923, 1936) experiments the cellular disturbances do not show continual amplification with the passage of time; rather a finite equilibrium amplitude is attained. It is clear that this effect is obscured by the theoretical linearization of the equations, and it is to be expected that non-linear amplitude effects will be important when the amplitude has grown to such values that linearization is invalid. The non-linear mechanics of such supercritical disturbances have been studied comprehensively by Stuart (1958, 1960*a*) and by Watson (1960). In the former of Stuart's papers, an energy-balance method was used to study the rotating cylinder problem in the special, but important, case when the outer cylinder is at rest and the inner one rotates, and the gap width is small. It was assumed that the fundamental of the disturbance was given spatially by linearized theory and that harmonics of the fundamental mode were unimportant. With the amplitude as an unspecified function of time, its equilibrium value was determined from the mean-motion equations and the energy integral of the disturbance, in terms of integrals of the spatial functions of linear stability theory. The mean motion is distorted by the Reynolds stress, and from the modified mean motion Stuart determined the torque required to maintain the motion; this was compared with the experimental observations of Taylor (1936).

For a wide range of the Taylor number above the critical value, the agreement between Stuart's values for the torque and those of Taylor is quite good. However, it is now known (Stuart 1960*b*) that even for a first approximation to the equilibrium amplitude one must take into account the generation of the harmonic of the fundamental and the distortion of the fundamental itself with regard to its radial dependence.

In this paper we develop a rigorous perturbation expansion for arbitrary gap width and any speeds of the containing cylinders, in order to determine the non-linear growth and equilibrium state of the vortices. The method requires the amplification rate of linearized theory to be sufficiently small. The problem is taken to be one with rotational symmetry, since unsymmetrical disturbances are important only at Taylor numbers rather larger than those considered here

(Coles 1960). For a sufficiently small amplification rate σ, the equilibrium amplitude A_e of the fundamental is given by an equation of the form

$$0 = \sigma A_e^2 + (k_1 + k_2 + k_3) A_e^4,$$

where k_1, k_2, k_3 are constants. Now k_1 essentially represents the energy transfer from the mean motion to the fundamental disturbance, due to distortion of the mean motion by the Reynolds stress; it is negative. The term containing k_2 represents transfer of energy from the fundamental to its first harmonic and the term k_3 represents the net energy transfer to the fundamental due to distortion of the fundamental with regard to its radial dependence. The torque required to maintain the motion also depends to first-order upon k_1, k_2 and k_3 and upon the eigenfunctions of linearized theory.

In §§ 5 and 7 numerical results obtained by using a digital computer are given for the three cases: (i) when the outer cylinder has twice the radius of the inner one and is kept at rest, (ii) when the gap is small and the cylinders rotate with nearly the same speeds, and (iii) when the gap is small and the outer cylinder is kept at rest. In the simplest case, (ii), the linear stability problem is governed by a sixth-order differential equation with constant coefficients. The computer was used to determine the eigenfunctions and higher-order functions associated with the disturbance, and from these calculations the values of k_1, k_2 and k_3 were obtained. In case (iii), when the outer cylinder is at rest, similar quantities are determined. It is clear from the results of (ii) and (iii), and from an examination of the terms in the relevant equations, that, in the small-gap problem, $k_1 + k_2 + k_3$ is almost proportional to $(1-m)^{-2}$ for any speed of the outer cylinder (provided it is in the same sense as that of the inner cylinder). Another interesting result is that in (iii) the harmonic of the fundamental derives nearly all its energy directly from the mean motion, rather than from the fundamental. Since it cannot exist without the fundamental it seems as though the fundamental plays the role of a 'catalyst'.

The theoretical results given in § 5 for case (i), the wide-gap problem, include a prediction of the additional torque required to maintain the vortices. Agreement with experimental results is very close, and is obtained over a far wider range of the Taylor number than one would expect, especially since, at these higher Taylor numbers, non-symmetric disturbances may be present. Presumably this is either numerically fortuitous, or occurs because the basic energetics of the flow change very slowly as the Taylor number is raised.

This additional torque is also predicted in § 7.2 for case (iii), the small-gap problem with the outer cylinder at rest. Good agreement is found with experimental results near to the critical speed. These results may be compared directly with those of Stuart (1958), who neglected the terms leading to k_2, k_3, determined the eigenfunctions with the approximation $m = 1$ from a variational procedure (Chandrasekhar 1953), and based his calculations on the neutral disturbance. When the cylinders rotate with nearly the same speeds, his method gives accurate results compared with the present calculation (when $m \to 1$); and when the outer cylinder is at rest, his result also is in good agreement with experiment and with the present theory.

2. Analysis of the basic equations

We use cylindrical co-ordinates (r, θ, z) and denote the corresponding velocity components by (u, v, w). We assume that the flow is axisymmetrical so that u, v, w are independent of θ. It is known from experimental work that the disturbance usually takes the form of cellular toroidal vortices spaced regularly along the axis of the cylinders; dependence on the azimuthal angle only occurs at higher Taylor numbers than those considered for any particular case in this paper.

The Navier–Stokes and continuity equations are

$$\frac{1}{R}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r} = -\frac{\partial p}{\partial r}+\frac{1}{R}\left(\nabla^2-\frac{1}{r^2}\right)u, \tag{2.1}$$

$$\frac{1}{R}\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r} = \frac{1}{R}\left(\nabla^2-\frac{1}{r^2}\right)v, \tag{2.2}$$

$$\frac{1}{R}\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z}+\frac{1}{R}\nabla^2 w, \tag{2.3}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z} = 0, \tag{2.4}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}. \tag{2.5}$$

In the above p denotes the pressure, R the Reynolds number and t the time. All quantities have been made dimensionless, the reference length being the distance $d = r_2 - r_1$ between the cylinders (the outer and inner cylinders having radii r_2, r_1 respectively), the reference velocity being $\Omega_1 r_1$ where Ω_1 is the angular velocity of the inner cylinder, the reference time being d^2/ν and the reference pressure $\rho\Omega_1^2 r_1^2$ where ρ is the fluid density (see Kirchgässner 1961). The Reynolds number $R = \Omega_1 r_1 d/\nu$, where ν is the kinematic viscosity.

The boundary conditions are

$$\left.\begin{array}{lll} u = v-1 = w = 0 & \text{when} & r = r_1/d, \\ u = v-m(1+d/r_1) = w = 0 & \text{when} & r = r_1/d+1, \end{array}\right\} \tag{2.6}$$

where $m = \Omega_2/\Omega_1$ is the ratio of the angular speeds of the outer and inner cylinders respectively.

For steady laminar Couette flow one has

$$v = A_0 r+\frac{B_0}{r}, \quad u = w = 0, \quad \frac{\partial p}{\partial r}-\frac{v^2}{r} = \frac{\partial p}{\partial z} = 0, \tag{2.7}$$

where

$$A_0 = \frac{d(1-mr_2^2/r_1^2)}{r_1(1-r_2^2/r_1^2)} \quad \text{and} \quad B_0 = \frac{r_1(1-m)}{d(1-r_1^2/r_2^2)}. \tag{2.8}$$

We will consider the growth of an infinitesimal disturbance which, as $t \to -\infty$, takes the form

$$u = u_1(r)\cos\lambda z\, e^{\sigma t}, \quad v = v_1(r)\cos\lambda z\, e^{\sigma t}, \quad w = w_1(r)\sin\lambda z\, e^{\sigma t}. \tag{2.9}$$

The linear stability problem (Chandrasekhar 1953) is then determined by

$$(DD^*-\lambda^2)(DD^*-\lambda^2-\sigma)^2 v_1 = 4A_0\,\Omega\lambda^2 R^2 v_1, \tag{2.10}$$

where
$$D \equiv \frac{d}{dr}, \quad D^* \equiv \frac{d}{dr}+\frac{1}{r}, \quad \Omega(r) \equiv A_0+\frac{B_0}{r^2}, \tag{2.11}$$

so that $\Omega(r)$ denotes the angular speed of the mean motion. The boundary conditions are

$$v_1 = DD^*v_1 = D(DD^*-\lambda^2-\sigma)\,v_1 = 0, \quad \text{when} \quad r = r_1/d \quad \text{and} \quad r = r_1/d+1. \tag{2.12}$$

These conditions determine a characteristic equation

$$F(\sigma, \lambda, R; r_2/r_1, \Omega_2/\Omega_1) = 0. \tag{2.13}$$

Thus, for fixed $R, r_2/r_1, \Omega_2/\Omega_1$ equation (2.13) determines σ for each wave-number λ. Experimental evidence supports the assumption that for R not too large and for all λ, σ is real, as we will assume, so that the disturbance takes the form of a non-oscillatory flow.

For a specific fluid with r_1, r_2, Ω_2 fixed, instability first occurs when Ω_1 attains a critical value at which $\sigma = 0$ for some wave-number λ. In general there is a denumerable number of (Ω_1, λ)-curves given by $\sigma = 0$, each corresponding to a different mode of instability. However, even when considering the growth of a finite disturbance it is the lowest mode which is most important and for given Ω_1, λ we take σ to be as given by perturbation from the lowest mode. For marginal stability ($\sigma = 0$), there is a unique wave-number λ which makes Ω_1 a minimum, that minimum being the critical value. In general (2.10) with the boundary conditions (2.12) is difficult to solve because Ω is not constant. The problem is simplified in cases where Ω may be satisfactorily approximated to by a constant.

The infinitesimal disturbance (2.9) satisfies the linear instability equations exactly and involves terms of the form $f(r,t)\cos\lambda z$ and $f(r,t)\sin\lambda z$. However, when the non-linear terms in (2.1), (2.2) and (2.3) are not neglected the disturbances react with themselves and the main flow, generating higher harmonics of the form

$$f_n(r,t)\,{}^{\cos}_{\sin}\,n\lambda z \quad (n = 2, 3, \ldots).$$

Thus it seems permissible to expand the disturbance velocity by using Fourier series. We take a representation of the form

$$u = u' = \sum_{n=1}^{\infty} u_n(r,t)\cos n\lambda z, \tag{2.14}$$

$$v = \bar{v}+v' = \bar{v}(r,t)+\sum_{n=1}^{\infty} v_n(r,t)\cos n\lambda z, \tag{2.15}$$

$$w = w' = \sum_{n=1}^{\infty} w_n(r,t)\sin n\lambda z. \tag{2.16}$$

In linearized theory the limit as $t \to -\infty$ of $\bar{v}(r,t)$ is the steady laminar solution and also as $t \to -\infty$ we have $u_n/u_1, v_n/v_1, w_n/w_1 \to 0$ for $n \geqslant 2$ and $u_1(r,t), v_1(r,t)$ and $w_1(r,t)$ tend to $u_1(r)\,e^{\sigma t}$, $v_1(r)\,e^{\sigma t}$ and $w_1(r)\,e^{\sigma t}$ respectively, where $u_1(r)$, $v_1(r)$ and $w_1(r)$ are the solutions (2.9). From (2.1) and (2.3) it follows that the pressure must be expressible in the form

$$p = \bar{p}+p' = \bar{p}(r,t)+\sum_{n=1}^{\infty} p_n(r,t)\cos n\lambda z. \tag{2.17}$$

The boundary conditions on the finite disturbance are that the mean velocity $\bar{v}$ takes the same values on the two cylinders as does the undisturbed velocity, that the disturbance velocities u', v' and w' vanish on the two cylinders, and that just enough external power is supplied to maintain the angular speeds of the cylinders at constant values, in accordance with the variation with time of the mean skin-friction on the cylinders.

Thus we must have

$$\left.\begin{array}{l} \bar{v} = 1 \quad \text{at} \quad r = r_1/d, \qquad \bar{v} = m(1+d/r_1) \quad \text{at} \quad r = r_1/d+1, \\ \left.\begin{array}{l} u_n = v_n = w_n = 0 \quad \text{at} \quad r = r_1/d, \\ u_n = v_n = w_n = 0 \quad \text{at} \quad r = r_1/d+1, \end{array}\right\} \quad (n = 1, 2, \ldots). \end{array}\right\} \tag{2.18}$$

Now substitute (2.14), (2.15), (2.16) in (2.1), (2.2), (2.3) and (2.4). The mean-motion equations obtained by equating terms which are independent of z are, as given by Stuart (1958),

$$\frac{1}{r}\frac{\partial}{\partial r}(r\overline{u'^2}) - \frac{1}{r}(\overline{v'^2} + \bar{v}^2) = -\frac{\partial \bar{p}}{\partial r}, \tag{2.19}$$

$$\frac{1}{R}\frac{\partial \bar{v}}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\overline{u'v'}) = \frac{1}{R}\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right\}\bar{v}, \tag{2.20}$$

where a bar above a quantity denotes a mean value with respect to z. The disturbance equations found by subtracting (2.19) from (2.1) and (2.20) from (2.2), and from (2.3) and (2.4) are

$$\frac{1}{R}\frac{\partial u'}{\partial t} - \frac{2\bar{v}v'}{r} + \chi_1 = -\frac{\partial p'}{\partial r} + \frac{1}{R}\left(\nabla^2 - \frac{1}{r^2}\right)u', \tag{2.21}$$

$$\frac{1}{R}\frac{\partial v'}{\partial t} + u'\left(\frac{\partial \bar{v}}{\partial r} + \frac{\bar{v}}{r}\right) + \chi_2 = \frac{1}{R}\left(\nabla^2 - \frac{1}{r^2}\right)v', \tag{2.22}$$

$$\frac{1}{R}\frac{\partial w'}{\partial t} + u'\frac{\partial w'}{\partial r} + w'\frac{\partial w'}{\partial z} = -\frac{\partial p'}{\partial z} + \frac{1}{R}\nabla^2 w', \tag{2.23}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru') + \frac{\partial w'}{\partial z} = 0, \tag{2.24}$$

where

$$\chi_1 = \frac{1}{r}\frac{\partial}{\partial r}(ru'^2 - r\overline{u'^2}) + \frac{\partial}{\partial z}(u'w') - \frac{(v'^2 - \overline{v'^2})}{r}, \tag{2.25}$$

$$\chi_2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2u'v' - r^2\overline{u'v'}) + \frac{\partial}{\partial z}(v'w'). \tag{2.26}$$

Using the Fourier expansions for u' and v' we may write the mean-motion equation (2.20) in the form

$$\frac{1}{R}\frac{\partial \bar{v}}{\partial t} + \sum_{n=1}^{\infty}\frac{1}{2r^2}\frac{\partial}{\partial r}(r^2 u_n v_n) = \frac{1}{R}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\bar{v}. \tag{2.27}$$

Now eliminate p' between (2.21) and (2.23) by differentiating the former with respect to z, the latter with respect to r and subtracting one from the other. Next, from the resulting equation, eliminate w' by using (2.24) and (2.16) and use the expansions (2.14) and (2.15) so that the nth ($n \geqslant 1$) component may be selected to yield

$$\mathscr{L}_1(n\lambda)\, u_n - 2n\lambda\bar{v}v_n/r = Q_1 + Q_2,\dagger \tag{2.28}$$

where Q_1 is a quadratic function of the u_i ($i \geqslant 1$) and Q_2 is a quadratic function of the v_i ($i \geqslant 1$), and where the operator $\mathscr{L}_1(n\lambda)$ is defined by

$$\mathscr{L}_1(n\lambda) \equiv (1/n\lambda R)\,(\mathscr{D}\mathscr{D}^* - n^2\lambda^2)\,(\mathscr{D}\mathscr{D}^* - n^2\lambda^2 - \partial/\partial t), \tag{2.29}$$

with

$$\mathscr{D} \equiv \partial/\partial r \quad \text{and} \quad \mathscr{D}^* \equiv (\partial/\partial r) + (1/r). \tag{2.30}$$

Moreover, we use the expansions (2.14) and (2.15) in equation (2.22) and select the nth ($n \geqslant 1$) component to give

$$(1/R)\,(\mathscr{D}\mathscr{D}^* - n^2\lambda^2 - \partial/\partial t)\, v_n - (\mathscr{D}^*\bar{v})\, u_n = Q_3 + Q_4 + Q_5, \tag{2.31}$$

where Q_3, Q_4 and Q_5 are linear functions of the quantities $u_i v_j$ with $i, j \geqslant 1$. Thus the problem now is to determine u_n, v_n and $\bar{v}$ from the infinite set of partial differential equations (2.28), (2.31) and (2.27).

3. Determination of disturbance growth

We are interested in the growth of a supercritical disturbance when the Taylor number is larger than its critical value. Thus we seek a solution of the equations of motion which represents a small finite disturbance, whose amplitude grows with time, with the property that as $t \to -\infty$ the disturbance tends through the infinitesimal disturbance of linear theory to zero.

Hence as $t \to -\infty$, we must have $u_1(r,t) \sim Cu_1(r)\, e^{\sigma t}$, $v_1(r,t) \sim Cv_1(r)\, e^{\sigma t}$, while $u_n \to 0$, $v_n \to 0$ ($n > 1$) more rapidly, where C is a constant. Thus we look for a solution in u_n, v_n which is separable and we suppose the highest-order terms in $u_1(r,t)$ and $v_1(r,t)$ to be of the form $A(t)\, u_1(r)$ and $A(t)\, v_1(r)$ respectively, where $A(t)$ is some, possibly bounded, function which behaves like $C\, e^{\sigma t}$ as $A \to 0$. We seek a solution in which A is small and which is such that $A^{-1}dA/dt$ is a function of A only. Thus we seek a solution for which

$$A^{-1}dA/dt = \sigma + \text{smaller-order terms.} \tag{3.1}$$

By setting

$$u_1(r,t) = A(t)\, u_1(r) + \text{smaller-order terms}$$

and

$$v_1(r,t) = A(t)\, v_1(r) + \text{smaller-order terms}$$

in (2.28) and (2.31) with $n = 1$, dividing by A and letting $A \to 0$ we obtain

$$(DD^* - \lambda^2)\,(DD^* - \lambda^2 - \sigma)\, u_1 - 2\lambda^2 R\bar{v}_l\, v_1/r = 0, \tag{3.2}$$

and

$$(DD^* - \lambda^2 - \sigma)\, v_1 - 2A_0 R u_1 = 0, \tag{3.3}$$

with D, D^* as defined by (2.11). These are the equations which determine the eigenfunctions $u_1(r)$ and $v_1(r)$ of linear theory and are equivalent to (2.10).

† Full details of Q_1, Q_2, and of Q_3, Q_4, Q_5 occurring in (2.31) may be found in the author's thesis (1961).

An investigation of (2.27), (2.28) and (2.31), similar to those of Stuart (1960*a*) and Watson (1960), suggests that we look for a solution of $\bar{v}$, u_n, v_n of the form

$$u_n(r,t) = A^n\left\{u_n(r) + \sum_{m=1}^{\infty} A^{2m} u_{nm}(r)\right\} \quad (n \geqslant 1), \tag{3.4}$$

$$v_n(r,t) = A^n\left\{v_n(r) + \sum_{m=1}^{\infty} A^{2m} v_{nm}(r)\right\} \quad (n \geqslant 1), \tag{3.5}$$

$$\bar{v} = \bar{v}_l + \sum_{m=1}^{\infty} A^{2m} f_m(r), \tag{3.6}$$

$$\frac{1}{A}\frac{dA}{dt} = \sum_{m=0}^{\infty} a_m A^{2m} \quad (a_0 = \sigma), \tag{3.7}$$

where a_m $(m \geqslant 1)$ are unknown constants; this leads to no inconsistency.

For the growth of our disturbance we are interested in the range of A between zero and the first non-zero positive root of the right-hand side of (3.7). The most important thing to do is to evaluate the constant a_1. Previous theoretical work by Stuart (1958) indicated that for the case when d/r_1 is small, a_1 is negative and non-zero. This is confirmed in § 7 of this paper, and the results of § 5 also indicate that a_1 is negative and non-zero when d/r_1 is not small. Provided the constants a_m $(m \geqslant 1)$ are bounded as $\sigma \to 0$ it is clear from (3.7) that, if A_e is the equilibrium amplitude, and is not large, then $A_e^2 = (-\sigma/a_1)\{1+O(A_e^2)\}$. In fact $(-\sigma/a_1)$ is a good approximation to A_e^2 for a wider range of Taylor numbers above the critical than one would expect, possibly because a_2 is much smaller than a_1. Thus evaluation of a_1 will tell us to order σ the value of A_e^2, and consequently, to the same order, the additional torque required to maintain the constant angular speeds of the cylinders. If we also calculate a_2 then we shall be able to determine A_e^2 and the torque correct to order σ^2 and such values, even for a wide gap, will possibly be quite accurate for a wide range of Taylor numbers above the critical.

We will now use the substitutions (3.4), (3.5), (3.6) and (3.7) primarily to obtain the equations for $u_n(r)$, $u_{nm}(r)$, $v_n(r)$, $v_{nm}(r)$, $f_m(r)$ which involve a_m but more specifically to obtain first u_1, v_1, f_1, u_2, v_2 followed by a_1 and u_{11}, v_{11}; and secondly to obtain f_2, u_3, v_3, u_{21}, v_{21} followed by a_2 and u_{12}, v_{12}.

From (2.28) and (2.31) with the substitutions (3.4)–(3.7), dividing both sides by their common factor A^n, and equating terms in each equation which are independent of A we obtain

$$L_1(n\lambda)\,u_n(r) - \frac{2n\lambda\bar{v}_l}{r}v_n(r) = \sum_{p=1}^{n-1}\left[\frac{n\lambda}{2r}\left\{v_p v_{n-p} - \frac{d}{dr}(ru_p u_{n-p}) + \frac{nu_p}{(n-p)}\frac{d}{dr}(ru_{n-p})\right\}\right.$$
$$\left. + \frac{1}{2\lambda(n-p)}\frac{d}{dr}\left\{u_p\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}(ru_{n-p})\right\} - \frac{1}{r^2}\frac{d}{dr}(ru_p)\frac{d}{dr}(ru_{n-p})\right\}\right], \tag{3.8}$$

and
$$\frac{1}{R}(DD^* - n^2\lambda^2 - n\sigma)\,v_n(r) - 2A_0 u_n(r)$$
$$= \sum_{p=1}^{n-1}\left[\frac{1}{2r^2}\frac{d}{dr}(r^2 u_p v_{n-p}) - \frac{nv_p}{2(n-p)\,r}\frac{d}{dr}(ru_{n-p})\right], \tag{3.9}$$

where
$$L_1(n\lambda) \equiv (1/n\lambda R)\,(DD^* - n^2\lambda^2)\,(DD^* - n^2\lambda^2 - n\sigma). \tag{3.10}$$

Balancing coefficients of A^{n+2} in the equations obtained from (2.28) and (2.31) we obtain

$$(1/n\lambda R)\,[(DD^*-n^2\lambda^2)\,(DD^*-n^2\lambda^2-(n+2)\,\sigma)]\,u_{n1} \\ -(2n\lambda/r)\,\{\bar{v}_l v_{n1}+v_n f_1\} = (a_1/\lambda R)\,(DD^*-n^2\lambda^2)\,u_n+g_{n1}(r), \quad (3.11)$$

and

$$R^{-1}(DD^*-n^2\lambda^2-(n+2)\,\sigma)\,v_{n1}-2A_0\,u_{n1} = (na_1/R)\,v_n+u_n\,D^*f_1+h_{n1}(r), \quad (3.12)$$

where $g_{nm}(r)$, $h_{nm}(r)$ are the coefficients of A^{2m+n} on the right-hand sides of (2.28) and (2.31) on using (3.4)–(3.7). In particular we shall need

$$4\lambda g_{11}(r) \equiv D(u_1\,DD^*u_2-2u_2DD^*u_1+D^*u_1\,D^*u_2) \\ +\lambda^2(4v_1\,v_2/r-4D^*(u_1\,u_2)+u_1\,D^*u_2-2u_2\,D^*u_1), \quad (3.13)$$

and

$$4h_{11}(r) \equiv (2/r^2)\,d\{r^2(u_1\,v_2+u_2\,v_1)\}/dr-v_1\,D^*u_2+2v_2\,D^*u_1. \quad (3.14)$$

Lastly we use the substitutions (3.4)–(3.7) in the mean-motion equation (2.27) and balancing coefficients of A^{2m} ($m \geqslant 1$), we obtain for $m = 1$

$$R^{-1}(DD^*-2\sigma)\,f_1 = (1/2r^2)\,d(r^2u_1\,v_1)/dr, \quad (3.15)$$

and for $m = 2$ that

$$R^{-1}[(DD^*-4\sigma)\,f_2-2a_1f_1] = (1/2r^2)\,d\{r^2(u_2\,v_2+2u_1\,v_1)\}/dr. \quad (3.16)$$

The boundary conditions on the new functions given in (3.4)–(3.6) follow from (2.18) and are

$$\left.\begin{aligned} &u_n = Du_n = u_{nm} = Du_{nm} = v_n = v_{nm} = f_m = 0, \\ \text{when}\quad &r = r_1/d \quad\text{and}\quad r = r_1/d+1 \quad (n = 1,2,\ldots;\ m = 1,2,\ldots). \end{aligned}\right\} \quad (3.17)$$

We may also obtain the equations for u_{nm}, v_{nm} ($m > 1$) by equating coefficients of A^{2m+n} in (2.28) and (2.31) on using (3.4)–(3.7) and the equations for f_m ($m > 2$) by equating coefficients of A^{2m} in the mean-motion equation (2.27) using (3.4)–(3.7). With the boundary conditions (3.17) we determine u_n, v_n, u_{n1}, v_{n1}, f_1 and a_1 from (3.8), (3.9), (3.11), (3.12) and (3.15), and u_{nm}, v_{nm}, f_m together with a_m ($m>1$) from the higher-order equations. Note that from (3.16) we may determine f_2 immediately a_1 is known. Fuller details of the equations for u_{nm}, v_{nm} and f_m are given in the author's thesis (1961).

4. Method of solution

We may eliminate u_1 between (3.2) and (3.3) to obtain

$$[(DD^*-\lambda^2)\,(DD^*-\lambda^2-\sigma)^2-4A_0\,\lambda^2R^2(A_0+B_0/r^2)]\,v_1 = 0, \quad (4.1)$$

where the boundary conditions on v_1 are

$$v_1 = DD^*v_1 = D(DD^*-\lambda^2-\sigma)\,v_1 = 0, \quad \text{at} \quad r = r_1/d \quad \text{and} \quad r = r_1/d+1. \quad (4.2)$$

Equation (4.1) with the boundary conditions (4.2) gives the eigenvalues and the eigenfunction v_1 determined to within an arbitrary multiplicative factor. To make v_1 definite we select that v_1 for which $v_1 = 1$ when $r = r_1/d+\frac{1}{2}$. In the event of an exceptional case with $v_1 = 0$ when $r = r_1/d+\frac{1}{2}$ we make v_1 definite by selecting that v_1 for which the integral of v_1^2 from r_1/d to $r_1/d+1$ is 1.

The value of u_1 is then given from (3.3) by

$$2A_0 Ru_1 = (DD^* - \lambda^2 - \sigma)\, v_1. \tag{4.3}$$

One may now determine u_2 and v_2 from (3.8) and (3.9) with $n = 2$. If we eliminate u_2 between these we find that v_2 satisfies

$$[L_1(2\lambda)(DD^* - 4\lambda^2 - 2\sigma) - 8A_0\lambda R(A_0 + B_0/r^2)]\, v_2 = \tfrac{1}{2}RL_1(2\lambda)(u_1 Dv_1 - v_1 Du_1)$$
$$+ 2A_0 R\left[\frac{\lambda}{r}(v_1^2 + u_1^2) + \frac{1}{2\lambda}\left\{u_1 u_1''' - u_1' u_1'' - \frac{u_1'^2}{r} - \frac{u_1 u_1''}{r} - \frac{3u_1 u_1'}{r^2} + \frac{4u_1^2}{r^3}\right\}\right], \tag{4.4}$$

where $L_1(2\lambda)$ is as defined by (3.10) and an accent denotes differentiation with respect to r. The boundary conditions satisfied by v_2 are

$$v_2 = DD^* v_2 = D(DD^* - 4\lambda^2 - 2\sigma)\, v_2 = 0, \quad \text{at} \quad r = r_1/d \quad \text{and} \quad r = r_1/d + 1. \tag{4.5}$$

The value of u_2 is then determined directly from

$$2A_0 Ru_2 = (DD^* - 4\lambda^2 - 2\sigma)\, v_2 - \tfrac{1}{2}R\{u_1 Dv_1 - v_1 Du_1\}. \tag{4.6}$$

In fact u_n and v_n for $n \geqslant 2$ may be found successively from (3.8) and (3.9) since the right-hand sides of these equations for $n = N$, say, are determined by u_i and v_i for $i \leqslant N - 1$.

Next we determine f_1 from equation (3.15) and the boundary conditions (3.17) with $m = 1$. We are now in a position to determine u_{11}, v_{11} and a_1 from (3.11) and (3.12) with $n = 1$. From these we may eliminate u_{11} and readily find that the important equation for v_{11} and a_1 is, using (4.3),

$$[(DD^* - \lambda^2)(DD^* - \lambda^2 - 3\sigma)^2 - 4A_0\lambda^2 R^2(A_0 + B_0/r^2)]\, v_{11}$$
$$= 2a_1(DD^* - \lambda^2)(DD^* - \lambda^2 - 2\sigma)\, v_1 + k_{11}(r), \tag{4.7}$$

where we define

$$\left.\begin{aligned} k_{11}(r) &\equiv k_{11}^{(1)}(r) + k_{11}^{(2)}(r),\\ k_{11}^{(1)}(r) &\equiv (4A_0\lambda^2 R^2/r)\, v_1 f_1 + R(DD^* - \lambda^2)(DD^* - \lambda^2 - 3\sigma).\, u_1 D^* f_1,\\ k_{11}^{(2)}(r) &\equiv R(DD^* - \lambda^2)(DD^* - \lambda^2 - 3\sigma)\, h_{11}(r) + 2A_0\lambda R^2 g_{11}(r). \end{aligned}\right\} \tag{4.8}$$

The boundary conditions to be satisfied by v_{11} are

$$\left.\begin{aligned} &v_{11} = DD^* v_{11} = D(DD^* - \lambda^2 - 3\sigma)\, v_{11} - a_1 Dv_1 = 0,\\ \text{at} \qquad &r = r_1/d \quad \text{and} \quad r = r_1/d + 1. \end{aligned}\right\} \tag{4.9}$$

Having found v_{11} and a_1 from (4.7) with (4.8) and (4.9) one may then determine u_{11} directly from

$$(DD^* - \lambda^2 - 3\sigma)\, v_{11} - 2A_0 Ru_{11} = a_1 v_1 + Ru_1 D^* f_1 + Rh_{11}(r). \tag{4.10}$$

(In (4.8) and (4.10) $g_{11}(r)$ and $h_{11}(r)$ are as given by (3.13) and (3.14), respectively.)

Equation (4.7) is a sixth-order differential equation for v_{11}, and the right-hand side is known completely save for the value of the constant a_1. To solve this consider first the case when $a_n = 0$ for $n \geqslant 1$, so that $A(t)$ is proportional to $e^{\sigma t}$. In general we wish to find a solution for A which is convergent and which is small at all times, and we expect to find such a solution for small values of σ.† Thus taking σ to be small we first solve (4.7) with $a_1 = 0$ that is

$$[(DD^* - \lambda^2)(DD^* - \lambda^2 - 3\sigma)^2 - 4A_0\lambda^2R^2(A_0 + B_0/r^2)]\,v_{11} = k_{11}(r), \quad (4.11)$$

subject to the boundary conditions (4.9) with $a_1 = 0$, namely

$$v_{11} = DD^*v_{11} = D(DD^* - \lambda^2 - 3\sigma)\,v_{11} = 0. \quad (4.12)$$

For small values of σ the square bracket on the left-hand side of (4.11) approximates very closely to the operator of linear theory occurring in (4.1). Moreover, the boundary conditions (4.12) differ only by terms of order σ from the boundary conditions (4.2). Thus the dominant term in v_{11} is probably a multiple of v_1. Moreover, on examining (4.11) and the boundary conditions (4.12) one readily sees that in general this multiple will tend to infinity as $\sigma \to 0$.

Thus we seek a solution of the form

$$v_{11} = \sigma^{-1}v_{11}^{(-1)} + v_{11}^{(0)} + \sigma v_{11}^{(1)} + \ldots, \quad (4.13)$$

where $v_{11}^{(-1)}, v_{11}^{(0)}, v_{11}^{(1)}, \ldots$, are bounded as $\sigma \to 0$. This method is due to Watson (1960). The numerical results of §§5 and 7 indicate that v_{11} will never have a multiple pole for any set of values of r_1, r_2 and m. Indeed one may show that if v_{11} had a double pole then σ would be purely imaginary on one side of the neutral curve.

On the functions $v_{11}^{(s)}$ we impose the following boundary conditions, which one can readily verify as being consistent with the boundary conditions on v_{11}, as given in (4.12) above:

$$v_{11}^{(-1)} = DD^*v_{11}^{(-1)} = D(DD^* - \lambda^2 - \sigma)\,v_{11}^{(-1)} = 0, \quad (4.14)$$

$$v_{11}^{(s)} = DD^*v_{11}^{(s)} = D(DD^* - \lambda^2 - \sigma)\,v_{11}^{(s)} - 2Dv_{11}^{(s-1)} = 0, \quad (4.15)$$

where (4.15) holds for all integral values of $s \geqslant 0$ and the boundaries at which (4.14) and (4.15) hold are at $r = r_1/d$ and $r = r_1/d + 1$.

The equations which are to be satisfied by the functions $v_{11}^{(s)}$ are

$$Lv_{11}^{(-1)} = 0, \quad (4.16)$$

$$Lv_{11}^{(0)} = 4Mv_{11}^{(-1)} + k_{11}(r), \quad (4.17)$$

$$Lv_{11}^{(s+1)} = 4Mv_{11}^{(s)} \quad (s \geqslant 0), \quad (4.18)$$

where we define

$$L \equiv (DD^* - \lambda^2)(DD^* - \lambda^2 - \sigma)^2 - 4A_0\lambda^2R^2(A_0 + B_0/r^2), \quad (4.19)$$

and

$$M \equiv (DD^* - \lambda^2)(DD^* - \lambda^2 - 2\sigma). \quad (4.20)$$

These equations, with (4.13), are easily verified as being consistent with (4.11).

The solution of (4.16) with the boundary conditions (4.14) is clearly $v_{11}^{(-1)} = \Lambda v_1$, where so far Λ is an arbitrary constant, so that the dominant term in v_{11} is $\Lambda v_1/\sigma$. Now substitute Λv_1 for $v_{11}^{(-1)}$ in the right-hand side of (4.17) to give a sixth-order

† It is suggested that since σ often appears in linear association with λ^2 a suitable criterion is $\sigma \ll \lambda^2$, but this has not yet been proved.

differential equation for $v_{11}^{(0)}$, the solution of which requires Λ to have a special value. To solve this we first determine Λ from

$$Lv_{11}^{(0)} = 4\Lambda M v_1 + k_{11}(r), \tag{4.21}$$

the boundary conditions being given by (4.15) with $s = 0$.

We define $\bar{L}$ to be the adjoint differential operator (Ince 1956) to the operator L, and it is readily shown that if we define

$$D^- \equiv (d/dr) - (1/r), \tag{4.22}$$

then
$$\bar{L} \equiv (D^-D - \lambda^2)(D^-D - \lambda^2 - \sigma)^2 - 4A_0\lambda^2R^2(A_0 + B_0/r^2). \tag{4.23}$$

The inhomogeneous boundary conditions on $v_{11}^{(0)}$ may be written, using (4.15) with $s = 0$ and $v_{11}^{(-1)} = \Lambda v_1$, as

$$\left.\begin{aligned} v_{11}^{(0)} = DD^*v_{11}^{(0)} = D(DD^* - \lambda^2 - \sigma)\,v_{11}^{(0)} - 2\Lambda D v_1 = 0, \\ \text{when} \quad r = r_1/d \quad \text{and} \quad r = r_1/d + 1. \end{aligned}\right\} \tag{4.24}$$

Now let $\theta(r)$† be the unique solution of $\bar{L}\theta = 0$ that satisfies the corresponding homogeneous adjoint boundary conditions, which are

$$\left.\begin{aligned} \theta = D\theta = (D^-D - \lambda^2)(D^-D - \lambda^2 - \sigma)\,\theta = 0, \\ \text{when} \quad r = r_1/d \quad \text{and} \quad r = r_1/d + 1. \end{aligned}\right\} \tag{4.25}$$

Hence if we multiply (4.21) throughout by θ and integrate from $r = r_1/d$ to $r = r_1/d + 1$ we obtain

$$\begin{aligned} \int_{r_1/d}^{r_1/d+1} \theta k_{11}\, dr + 4\Lambda \int_{r_1/d}^{r_1/d+1} \theta M v_1\, dr &= \int_{r_1/d}^{r_1/d+1} \theta L v_{11}^{(0)}\, dr \\ &= \int_{r_1/d}^{r_1/d+1} v_{11}^{(0)} \bar{L}\theta\, dr + 2\Lambda[Dv_1\, D^2\theta]_{r_1/d}^{r_1/d+1}. \end{aligned} \tag{4.26}$$

Thus, since $\bar{L}\theta = 0$, equation (4.26) determines Λ and we may write

$$\int_{r_1/d}^{r_1/d+1} \theta k_{11}\, dr + 4\Lambda \int_{r_1/d}^{r_1/d+1} \theta M v_1\, dr = 2\Lambda[Dv_1\, D^2\theta]_{r_1/d}^{r_1/d+1}. \tag{4.27}$$

Now that Λ is known the right-hand side of (4.21) is determined and we may solve this equation for $v_{11}^{(0)}$. One method, after Watson (1960), is to define functions $\chi_2, \chi_3, \chi_4, \chi_5, \chi_6$ which are solutions of $L(\chi) = 0$, the corresponding homogeneous equation, so that $v_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6$ are linearly independent solutions. Thus $v_{11}^{(0)}$ is a linear combination of these functions (apart from any particular integral of (4.21)), and by using (4.18) with $s = 0, 1, 2, \ldots$, we may determine v_{11} completely when $a_1 = a_2 = \ldots = a_n = 0$.

This solution corresponds to $A(t)$ being proportional to $e^{\sigma t}$ and does not converge as $t \to \infty$. To find a solution which converges for all time we must first choose a suitable value for a_1 and then suitable values for $a_2, a_3, \ldots, a_n$, in turn.

† The quantity $\theta(r)/r$ satisfies the boundary conditions on the radial eigenfunction u_1 and the same equation, $L[\theta(r)/r] = 0$, as the azimuthal eigenfunction v_1. Thus if v_g is the general solution of $Lv = 0$ and θ_g is the general solution of $\bar{L}\theta = 0$, then $\theta_g = rv_g$ which is a circulation expression.

The solution, v_{11}, of (4.7) subject to the boundary conditions (4.9) consists of the solution of (4.11) just obtained together with, from (4.19) and (4.20), the solution of

$$Lv_{11} - 4\sigma M v_{11} = 2a_1 M v_1, \tag{4.28}$$

subject to the boundary conditions (4.9). Clearly the solution of (4.28) subject to the boundary conditions (4.9) is $-(a_1/2\sigma)v_1$. Thus the full solution of (4.7) is

$$v_{11} = \sigma^{-1}(\Lambda - \tfrac{1}{2}a_1)v_1 + O(1), \tag{4.29}$$

and if we choose a_1 so that $\Lambda - \frac{1}{2}a_1 = 0$ then v_{11} will be bounded as $\sigma \to 0$. Thus to make the series for $v_1(r,t)$ converge as rapidly as possible, a_1 is determined by $a_1 = 2\Lambda$, and thence

$$v_{11} = v_{11}^{(0)} + \sigma v_{11}^{(1)} + \ldots. \tag{4.30}$$

With a_1 and v_{11} thus determined, u_{11} follows from (4.10) and v_{n1}, u_{n1} are found from (3.11) and (3.12); moreover f_2 may be found from (3.16).

At this stage $g_{12}(r)$, $h_{12}(r)$ are known functions. One may readily show that v_{12} is expandable when $a_2 = 0$ in the same way as v_{11}, and that a_2 may be chosen to remove the simple pole in v_{12}. Then u_{12} can be determined. Moreover v_{n2}, u_{n2} follow successively (g_{n2}, h_{n2} being known when the equations for u_{n2}, v_{n2} are to be solved) and f_3, after which v_{13}, u_{13} and a_3 can be similarly determined. We may proceed in this way successively to find the constants a_m provided $m\sigma$ is very small. When $m\sigma$ is not small the equation for v_{1m} is no longer ill-conditioned and the series expansion loses its accuracy. Thus for $m\sigma$ not small no particular value of a_m will make the series for $v(r,t)$ converge more rapidly than any other value and we may set $a_m = 0$ for all sufficiently large m. Proceeding in this way all the functions of r and the constants a_m appearing in (3.7) may be found.

Now we may determine to order σ the equilibrium amplitude by retaining only the first two terms on the right-hand side of equation (3.7), provided that v_{11} does not have a multiple pole. We have on this assumption that

$$dA/dt = \sigma A + a_1 A^3,$$

and the solution which satisfies $A \sim C\,e^{\sigma t}$ as $A \to 0$ is

$$A^2 = K\sigma\,e^{2\sigma t}/(1 - a_1 K\,e^{2\sigma t}), \tag{4.31}$$

where $C^2 = K\sigma$, their values depending upon the origin of t. Hence to a first approximation the equilibrium amplitude A_e is given by

$$A_e^2 = -\sigma/a_1, \tag{4.32}$$

so that A_e^2 is of order σ. According as $a_1 > 0$ or $a_1 < 0$, we will have subcritical or supercritical disturbances which will decay from or amplify up to their equilibrium values respectively. The experimental work of Taylor (1936) and Donnelly (1958) indicates that only supercritical disturbances occur so that we expect $a_1 < 0$ when the outer cylinder is held at rest, and also when the cylinders rotate in the same or opposite directions. Experimental evidence suggests that A_e^2 is proportional to $T - T_c$ for small values of this quantity, but this sheds no light on the multiplicity of the pole in v_{11}.† Here T is the Taylor number defined in (5.2) below and T_c its 'critical' value. However the numerical calculations of §§ 5 and 7 indicate that the limit as $\sigma \to 0$ of the coefficient of Λ in (4.27) will be zero for at most a few discrete sets of values of Ω_1, Ω_2, r_1 and d. It is only in these unlikely cases that we need resort to a double- or higher-pole expansion in (4.13). In such a case the functions $v_{11}^{(-p+s)}$ with $s \geqslant 0$ must be defined so that they do not depend upon σ.

5. The wide-gap problem with $\Omega_2 = 0$, $r_2 = 2r_1$

The most important properties one wishes to determine in the solution of the non-linear stability problem are the shape and strength of the Taylor vortices. Here we consider the particular case when the outer cylinder is at rest and has a radius which is twice that of the inner cylinder. The solution we obtain is an exact solution of the Navier–Stokes equations under the limiting condition $\sigma \to 0$, which implies that the Taylor number only slightly exceeds its critical value. In fact this restriction is not as strong as might appear, since the analysis gives very good results for a surprisingly wide range of the Taylor number above this value.

Using $m = 0$ and $r_2 = 2r_1$ in (2.8) one obtains

$$A_0 = -\tfrac{1}{3}, \quad B_0 = \tfrac{4}{3}, \tag{5.1}$$

and in this special problem we define a Taylor number by

$$T \equiv \tfrac{64}{9}R^2. \tag{5.2}$$

Using (5.1), (5.2) in (4.1) the linear stability problem is specified by

$$[(DD^* - \lambda^2)(DD^* - \lambda^2 - \sigma)^2 + \lambda^2 T(\tfrac{1}{4}r^{-2} - \tfrac{1}{16})]\, v_1 = 0, \tag{5.3}$$

with the boundary conditions from (4.2) as

$$v_1 = DD^* v_1 = D(DD^* - \lambda^2 - \sigma)\, v_1 = 0 \quad \text{at} \quad r = 1, 2. \tag{5.4}$$

The shape of the vortices will depend on the Taylor number and we determine this shape in the limit as $T - T_c \to 0$, where T_c is the critical Taylor number. When $T - T_c$ is small, σ the amplification rate of the corresponding infinitesimal disturbances is also small, and so is the equilibrium amplitude of the vortices. Thus we content ourselves with finding the limiting form of the vortices as $\sigma \to 0$, as given by the limits as $\sigma \to 0$ of u_1, v_1 and u_2, v_2. These are sufficient to calculate the limit as $\sigma \to 0$ of the constant a_1 from formula (4.27) and the relation $a_1 = 2\Lambda$, provided we also determine the limiting form as $\sigma \to 0$ of the adjoint function θ.

Experimental evidence supports the assumption that λ may be fixed at the value λ_c which makes T a minimum when $\sigma = 0$. At a higher Taylor number, the value of λ which makes σ a maximum is only slightly greater than λ_c, and the variation of σ with λ is also only very small. Moreover, if the wavelength were to alter, the vortices would have to move axially and new ones would have to form at the ends of the cylinders. A fixed value of λ avoids this difficulty.

† For if v_{11} has a pole of order $p \geqslant 1$ one may show that Λ or a_1 is of order σ^{1-p} and that $T - T_c$ is proportional to σ^p, so that in all cases A_e^2 is proportional to $T - T_c$.

We started by using a digital computer to determine λ, T_c together with the eigenfunction v_1 and its derivatives in the limit as $\sigma \to 0$, using (5.3), (5.4) with $\sigma = 0$ and $T = T_c$. To do this we fixed λ at some value and set $v_1'(1) = 1$, so that the boundary conditions become $v_1(1) = 0$, $v_1''(1) = -1$ and $v_1'''(1) = \lambda^2+3$. Then we used an iterative procedure which, starting with arbitrary values of T, $v_1^{(iv)}(1)$, $v_1^{(v)}(1)$, converged to a set of these values which made $v_1(2) = 0$, $v_1''(2)+\frac{1}{2}v_1'(2) = 0$ and $v_1'''(2)-\lambda^2 v_1'(2)-\frac{3}{4}v_1'(2) = 0$ on integrating across the gap between the cylinders. The value of λ was then found which made T a minimum and the eigenfunction was normalized to make $v_1(1{\cdot}5) = 1$. We found that T has a minimum value of $T_c = 33062$ when $\lambda = 3{\cdot}163$, and we shall now fix λ at this value.

For convenience we now define

$$\bar{u}_1 \equiv 2A_0Ru_1, \quad F_1 \equiv 4A_0f_1, \quad \bar{v}_2 \equiv 4A_0v_2, \quad \bar{u}_2 \equiv 8A_0^2Ru_2, \tag{5.5}$$

and determine the limiting forms of $\bar{u}_1$, F_1, $\bar{v}_2$, $\bar{u}_2$ and their derivatives as $\sigma \to 0$. To determine $\bar{u}_1$ put $\sigma = 0$ in (4.3) and use (5.5) to obtain

$$\bar{u}_1 = (DD^*-\lambda^2)\,v_1, \tag{5.6}$$

which, since v_1 and its derivatives are now known yields the values of $\bar{u}_1$ and its derivatives (with successive differentiations of (5.6) and use of (5.3) with $\sigma = 0$ and $T = T_c$). The values of v_1, $\bar{u}_1$ and their first three derivatives are to be found in table 1 which has been deposited with the Editor.†

Next we found the limiting form of the adjoint function θ. By putting $\sigma = 0$ in (4.23) and using $\bar{L}\theta = 0$ with $T = T_c$ we have

$$[(D^-D-\lambda^2)^3+\lambda^2T_c(\tfrac{1}{4}r^{-2}-\tfrac{1}{16})]\,\theta = 0; \tag{5.7}$$

the relevant boundary conditions, found from (4.25) are

$$\theta = D\theta = (D^-D-\lambda^2)^2\,\theta = 0 \quad \text{at} \quad r = 1, 2. \tag{5.8}$$

The magnitude of θ is not important and for simplicity we choose $\theta''(1) = 1$. Then θ and its derivatives were found using the computer by integrating from $r = 1$ to $r = 2$. The quantities $\theta'''(1)$, $\theta^{(v)}(1)$ were chosen so that $\theta(2) = \theta'(2) = 0$, while the third boundary condition was satisfied with an error of less than 1 part in 10^5. The function θ and its first three derivatives are to be found in table 2.

Now we may determine the relationship between σ and T in the limit as $\sigma \to 0$. The equation for $v \equiv v_1(r;\sigma)$ is (5.3) with the boundary conditions (5.4). Since σ is small let $v = v_1+\sigma\hat{v}+O(\sigma^2)$, $T = T_c+\sigma\epsilon+O(\sigma^2)$ where ϵ is a constant to be determined and where now $v_1 \equiv v_1(r;0)$. To zero order in σ the boundary conditions on v_1, $\hat{v}$ are

$$\left.\begin{aligned} v_1 = DD^*v_1 = D(DD^*-\lambda^2)\,v_1 = 0,\\ \hat{v} = DD^*\hat{v} = D(DD^*-\lambda^2)\,\hat{v}-Dv_1 = 0,\end{aligned}\right\} \quad \text{at} \quad r = 1, 2, \tag{5.9}$$

and the equation for $\hat{v}$ is

$$[(DD^*-\lambda^2)^3+\lambda^2T_c(\tfrac{1}{4}r^{-2}-\tfrac{1}{16})]\,\hat{v} = 2(DD^*-\lambda^2)^2\,v_1-\epsilon\lambda^2(\tfrac{1}{4}r^{-2}-\tfrac{1}{16})\,v_1. \tag{5.10}$$

† Tables 1 to 11 inclusive have been lodged with the Editor of the *Journal of Fluid Mechanics* and may be consulted by readers on application to the Editor. Tables A and B given in the text summarize the more important results.

The unique value of ϵ which permits (5.10) to have a solution was found using the theory of § 4; in particular the results (4.26), (4.27) are required in the special case $2\Lambda = 1$. We also need the special result

$$\int_1^2 \theta(DD^* - \lambda^2)^2 v_1 \, dr - [\theta'' v_1']_1^2 = \int_1^2 v_1(D^- D - \lambda^2)^2 \theta \, dr, \tag{5.11}$$

which can be derived by integration by parts. Multiply (5.10) throughout by $\theta(r; 0)$ and integrate over the gap between the cylinders. Using the boundary conditions (5.9), together with (5.7), (5.8), (5.11) we find that

$$\frac{1}{\epsilon} = \frac{\lambda^2 \int_1^2 \theta\left(\frac{1}{4r^2} - \frac{1}{16}\right) v_1 \, dr}{\left[\int_1^2 \theta(DD^* - \lambda^2)^2 v_1 \, dr + \int_1^2 v_1(D^- D - \lambda^2)^2 \theta \, dr\right]}. \tag{5.12}$$

We evaluated the integrals in (5.12) on the computer and found

$$\epsilon^{-1} = 4{\cdot}0645 \times 10^{-4}$$

so that

$$\sigma = 13{\cdot}44(1 - T_c/T).\dagger \tag{5.13}$$

Next we found F_1, which measures the distortion of the mean motion by the Reynolds stress. In (3.15) put $\sigma = 0$ and use (5.5) to obtain

$$DD^* F_1 = D(\bar{u}_1 v_1) + 2\bar{u}_1 v_1/r, \tag{5.14}$$

with the boundary conditions $F_1 = 0$ at $r = 1, 2$. The computer was used to determine F_1 and its derivative by integrating from $r = 1$ to $r = 2$. The quantity $F_1'(1)$ was chosen so that $F_1(2) = 0$ and the function and its derivative are to be found in table 3.

Next we found $\bar{v}_2$ and its derivatives as $\sigma \to 0$. We use (4.4), (5.1) and (5.5) with $\sigma = 0$ to obtain

$$\left[(DD^* - 4\lambda^2)^3 + 4\lambda^2 T_c\left(\frac{1}{4r^2} - \frac{1}{16}\right)\right] \bar{v}_2 = (DD^* - 4\lambda^2)^2 (\bar{u}_1 v_1' - \bar{u}_1' v_1)$$
$$+ 2\left[\bar{u}_1 \bar{u}_1''' - \bar{u}_1' \bar{u}_1'' - \frac{\bar{u}_1'^2}{r} - \frac{\bar{u}_1 \bar{u}_1''}{r} - \frac{3\bar{u}_1 \bar{u}_1'}{r^2} + \frac{4\bar{u}_1^2}{r^3} + \frac{2\lambda^2}{r}\left(\frac{T_c v_1^2}{16} + \bar{u}_1^2\right)\right]; \tag{5.15}$$

for ease of computation we used (5.6) to rewrite the right-hand side of (5.15) in terms of v_1. The appropriate boundary conditions obtained from (4.5) are

$$\bar{v}_2 = DD^* \bar{v}_2 = D(DD^* - 4\lambda^2)\, \bar{v}_2 = 0 \quad \text{at} \quad r = 1, 2. \tag{5.16}$$

Since the right-hand side of (5.15) is known, $\bar{v}_2$ and its derivatives were found by integrating from $r = 1$ to $r = 2$. A programme was written which, given arbitrary initial values of $\bar{v}_2'''(1)$, $\bar{v}_2^{(\mathrm{iv})}(1)$ and $\bar{v}_2^{(\mathrm{v})}(1)$, converged to a set of these values which made $\bar{v}_2(2)$, $\bar{v}_2''(2) + \frac{1}{2}\bar{v}_2'(2)$ and $\bar{v}_2'''(2) - \frac{3}{4}\bar{v}_2'(2) - 4\lambda^2 \bar{v}_2'(2)$ all zero.

† This form of the relationship between σ and T is valid to first order in $T - T_c$. Moreover it will be seen later that the use of the form (5.13) to calculate A_e^2 from (4.32) often yields a result in good agreement with experiment, even for ranges of T for which (5.13) is not valid.

The last function required is $\bar{u}_2$ with $\sigma = 0$ and this is given directly from (4.6) with $\sigma = 0$ and (5.5) to obtain

$$\bar{u}_2 = (DD^* - 4\lambda^2)\,\bar{v}_2 + (v_1\bar{u}_1' - v_1'\bar{u}_1). \tag{5.17}$$

From the knowledge of $\bar{v}_2$, v_1, $\bar{u}_1$ and their derivatives and from successive differentials of (5.17) we also found the derivatives of $\bar{u}_2$. The functions $\bar{v}_2$, $\bar{u}_2$ and their first three derivatives are to be found in table 4.

r	v_1	$10^{-1}\bar{u}_1$	$10^2\theta$	F_1	$\bar{v}_2$	$10^{-1}\bar{u}_2$
1·00	0·000	0·000	0·000	0·000	0·000	0·000
1·05	0·175	−0·126	0·114	0·894	−0·212	0·272
1·10	0·343	−0·428	0·413	1·713	−0·418	0·934
1·15	0·503	−0·817	0·837	2·397	−0·607	1·803
1·20	0·648	−1·221	1·329	2·879	−0·766	2·738
1·25	0·775	−1·592	1·838	3·109	−0·885	3·626
1·30	0·878	−1·896	2·322	3·069	−0·960	4·380
1·35	0·954	−2·113	2·742	2·779	−0·991	4·942
1·40	0·999	−2·234	3·069	2·290	−0·982	5·278
1·45	1·015	−2·258	3·282	1·675	−0·940	5·382
1·50	1·000	−2·192	3·365	1·016	−0·872	5·269
1·55	0·958	−2·046	3·316	0·391	−0·785	4·965
1·60	0·891	−1·836	3·135	−0·137	−0·686	4·506
1·65	0·805	−1·577	2·833	−0·526	−0·583	3·931
1·70	0·702	−1·288	2·431	−0·761	−0·479	3·278
1·75	0·589	−0·987	1·953	−0·845	−0·379	2·582
1·80	0·469	−0·693	1·435	−0·800	−0·286	1·879
1·85	0·348	−0·426	0·921	−0·660	−0·202	1·208
1·90	0·228	−0·206	0·464	−0·459	−0·127	0·617
1·95	0·112	−0·056	0·131	−0·232	−0·060	0·178
2·00	0·000	0·000	0·000	0·000	0·000	0·000

TABLE A. Summary of results for the wide-gap problem $r_2 = 2r_1$, $m = 0$ with $v_1(1{\cdot}5) = 1$ and $\theta''(1) = 1$.

We are now in a position to find the limiting value as $\sigma \to 0$ of the constant a_1. For convenience we define

$$\bar{v}_{11} \equiv 8A_0^2 v_{11}, \quad \bar{u}_{11} \equiv 16A_0^3 R u_{11}, \quad \bar{g}_{11} \equiv 16A_0^3 R^2 g_{11}, \quad \bar{h}_{11} \equiv 8A_0^2 R h_{11}, \tag{5.18}$$

and

$$\bar{a}_1 \equiv 8A_0^2 a_1. \tag{5.19}$$

Using $2\Lambda = a_1$ and $\sigma = 0$ together with the formulae (4.8), (4.27), (5.5), (5.11), (5.18) and (5.19), and performing some integrations by parts, we have

$$-\bar{a}_1\left[\int_1^2 \theta(DD^* - \lambda^2)^2 v_1\,dr + \int_1^2 v_1(D^-D - \lambda^2)^2\,\theta\,dr\right]$$
$$= \frac{\lambda^2 T_c}{8}\int_1^2 \frac{\theta v_1 F_1}{r}\,dr + \int_1^2 \bar{u}_1\,D^*F_1(D^-D - \lambda^2)^2\,\theta\,dr + \int_1^2 [\lambda\theta\bar{g}_{11} + \bar{h}_{11}(D^-D - \lambda^2)^2\,\theta]\,dr, \tag{5.20}$$

where (3.13), (3.14), (5.1), (5.5) and (5.18) yield

$$4\lambda\bar{g}_{11} \equiv \tfrac{1}{4}\lambda^2 T_c v_1 \bar{v}_2 r^{-1} - \lambda^2[3\bar{u}_1\bar{u}_2' + 6\bar{u}_1'\bar{u}_2 + 5\bar{u}_1\bar{u}_2 r^{-1}] + [\bar{u}_1\bar{u}_2''' + 2\bar{u}_1'\bar{u}_2'' - \bar{u}_1''\bar{u}_2' - 2\bar{u}_1'''\bar{u}_2 + (2\bar{u}_1\bar{u}_2'' + \bar{u}_1'\bar{u}_2' - \bar{u}_1''\bar{u}_2) r^{-1} + 3\bar{u}_1'\bar{u}_2 r^{-2} - 4\bar{u}_1\bar{u}_2 r^{-3}], \quad (5.21)$$

and

$$4\bar{h}_{11} \equiv 4\bar{u}_1'\bar{v}_2 + 2\bar{u}_1\bar{v}_2' + 2v_1'\bar{u}_2 + v_1\bar{u}_2' + 3(2\bar{u}_1\bar{v}_2 + v_1\bar{u}_2) r^{-1}. \quad (5.22)$$

Evaluating the integrals in (5.20) we have

$$\bar{a}_1 = -132{\cdot}47 \quad \text{and} \quad \bar{a}_{12} = -27{\cdot}00. \quad (5.23)$$

(The latter value $\bar{a}_{12}$ is given here for later convenience and is the contribution to $\bar{a}_1$ of the harmonic terms represented by the last integral on the right-hand side of (5.20).) Hence, as $A_e^2 = -\sigma/a_1$, the square of the equilibrium amplitude of the vortices, found by using (5.13), (5.19) and (5.23), is given by

$$A_e^2 = 0{\cdot}09017(1 - T_c/T), \quad (5.24)$$

with the normalizing condition $v_1(1{\cdot}5) = 1$.

Having determined a_1, we now show that the differential equation which governs the disturbance amplitude is in fact an energy-balance relation for the fundamental disturbance (u_1, v_1, w_1). If we define u', v', w' to represent the velocity components of that part of the disturbance which has odd wave-numbers $(\lambda, 3\lambda, 5\lambda, \ldots)$, and u'', v'', w'' to represent the velocity components of that part of the disturbance which has even wave-numbers $(2\lambda, 4\lambda, 6\lambda, \ldots)$, then it can be shown from (2.21), (2.22) and (2.23) that

$$\frac{1}{R}\frac{\partial}{\partial t}\iint \tfrac{1}{2}(u'^2 + v'^2 + w'^2)\, r\, dr\, dz = \iint (-u'v')\left(\frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r}\right) r\, dr\, dz - \frac{1}{R}\iint (\xi'^2 + \eta'^2 + \zeta'^2)\, r\, dr\, dz - \iint (u'\chi_{11} + v'\chi_{21} + w'\chi_{31})\, r\, dr\, dz. \quad (5.25)$$

In (5.25) the integration ranges over one wavelength $(2\pi/\lambda)$ and between the cylinders. Also

$$\xi' = -\frac{\partial v'}{\partial z}, \quad \eta' = \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial r}, \quad \zeta' = \frac{1}{r}\frac{\partial}{\partial r}(rv') \quad (5.26)$$

are the vorticity components, and $\chi_{11}, \chi_{12}, \chi_{31}$ are the 'odd' parts of χ_1, χ_2, χ_3 in (2.25) and (2.26), and the non-linear part of the left-hand side of (2.23); thus

$$\chi_{11} \equiv r^{-1}\,\partial(2ru'u'')/\partial r + \partial(u'w'' + u''w')/\partial z - 2v'v''/r, \quad (5.27)$$

$$\chi_{21} \equiv r^{-2}\,\partial(r^2u'v'' + r^2u''v')/\partial r + \partial(v'w'' + v''w')/\partial z, \quad (5.28)$$

$$\chi_{31} \equiv u'\partial w''/\partial r + u''\partial w'/\partial r + \partial(w'w'')/\partial z. \quad (5.29)$$

Equation (5.25) states that the rate of increase of energy of the 'odd' part of the disturbance (u', v', w') equals the rate of transfer of energy from the mean motion, less the rate of dissipation of energy, less the rate of transfer of energy from the 'odd' to the 'even' (u'', v'', w'') part of the disturbance. By substituting (2.14), (2.15), (2.16) and also (3.4), (3.5), (3.6) in (5.25) one obtains

$$\tfrac{1}{2}\,dA^2/dt = \sigma A^2 + (k_1 + k_2 + k_3)\,A^4, \quad (5.30)$$

where some terms of order σ in the coefficient of A^4 have been ignored (together with higher powers of A^2).

From (5.25) it is readily shown that

$$k_1 = -\frac{1}{8A_0^2 k_0}\int_1^2 \bar{u}_1 v_1 (D^- F_1)\, r\, dr, \tag{5.31}$$

$$\begin{aligned} k_2 = -\frac{1}{8A_0^2 k_0}\Bigg[&\frac{8}{T_c}\int_1^2 \bar{u}_1\left(\frac{\bar{u}_1\bar{u}_2}{r}-\frac{T_c v_1 \bar{v}_2}{8r}\right) r\, dr \\ &+\int_1^2 v_1\left\{\bar{u}_1'\bar{v}_2+\tfrac{1}{2}\bar{u}_1\bar{v}_2'+\frac{1}{2r}(3\bar{u}_1\bar{v}_2+\bar{u}_2 v_1)\right\} r\, dr+\frac{4}{\lambda^2 T_c}\int_1^2\left(\bar{u}_1'+\frac{\bar{u}_1}{r}\right) \\ &\times\left\{\bar{u}_1\bar{u}_2''+\bar{u}_1'\bar{u}_2'-2\bar{u}_1''\bar{u}_2+\frac{1}{r}(2\bar{u}_1\bar{u}_2'-\bar{u}_1'\bar{u}_2)+\frac{2\bar{u}_1\bar{u}_2}{r^2}\right\} r\, dr\Bigg], \end{aligned} \tag{5.32}$$

$$k_3 = -(1/8A_0^2 k_0)\, Z, \tag{5.33}$$

where

$$k_0 = \int_1^2\left\{\frac{16\bar{u}_1^2}{T_c}+v_1^2+\frac{16(D^*\bar{u}_1)^2}{\lambda^2 T_c}\right\} r\, dr, \tag{5.34}$$

and Z is an integral whose integrand is a function of $\bar{u}_1$, v_1, $\bar{u}_{11}$ and $\bar{v}_{11}$.

From the above expressions for k_1, k_2, k_3 which are evaluated with $\sigma = 0$ it is clear from a comparison of (3.7) and (5.30) that the limit as $\sigma \to 0$ of a_1 is $k_1+k_2+k_3$ so that this sum is known from (5.23). As a consistency check (5.30) with the same expressions for k_1, k_2, k_3 may also be obtained directly from (3.7) and (3.11), (3.12) with $n = 1$. This is done by multiplying (3.12, $n = 1$) by Rv_1, multiplying (3.11, $n = 1$) by $2A_0\lambda R^2\bar{u}_1$, subtracting and integrating from $r = 1$ to $r = 2$.

Now $8A_0^2 k_1$ and $8A_0^2 k_2$ may be evaluated directly from (5.31), (5.32) and, since $a_1 = k_1+k_2+k_3$, we may use (5.23) to determine k_3. Since the equilibrium amplitude depends so strongly on the value of a_1, it is instructive to investigate the signs and relative magnitudes of k_1, k_2 and k_3. These quantities represent the following three physical processes: (i) the distortion of the mean motion (k_1); (ii) the generation of the harmonic of the fundamental (k_2); (iii) the distortion of the fundamental, with regard to its dependence on the radial co-ordinate (k_3). A schematic diagram of the energy supply to and from the fundamental and the harmonic to order A^4 is shown in figure 1. We found that

$$8A_0^2 k_1 = -113{\cdot}22, \quad 8A_0^2 k_2 = -16{\cdot}66, \quad 8A_0^2 k_3 = -2{\cdot}59, \tag{5.35}$$

and also $8A_0^2 k_{32} = -10{\cdot}34$ where k_{32} is the contribution of the harmonic to k_3. As expected k_1 is negative; it represents flow of energy to the disturbance from the mean motion. Also k_2 is negative, though much smaller than k_1, so that the harmonic extracts energy from the fundamental. Now k_3/k_1 is very small, so that the distortion of the fundamental has little effect on the equilibrium amplitude. However we may write $k_3 = k_{31}+k_{32}$, where k_{31} is the effect of the mean motion on the distortion of the fundamental, so it is clear from (5.35) that the two effects tending to distort the fundamental are separately appreciable. By considering an equation similar to (5.25) for the 'even' part of the disturbance and evaluating some integrals we may show that the harmonic actually extracts twice as much energy from the mean motion as is supplied to it by the fundamental, all being lost in viscous dissipation.

It is interesting to note that in the small-gap problem described in §§ 6 and 7, the mechanics is somewhat different.

Comparison with experiment for $r_2 = 2r_1$, $m = 0$

Now that the amplitude of the vortices is known, together with the distortion of the mean motion, we may calculate the torque required to maintain the motion. This is greater than the laminar value because of the vortices, and may be determined by experimental methods.

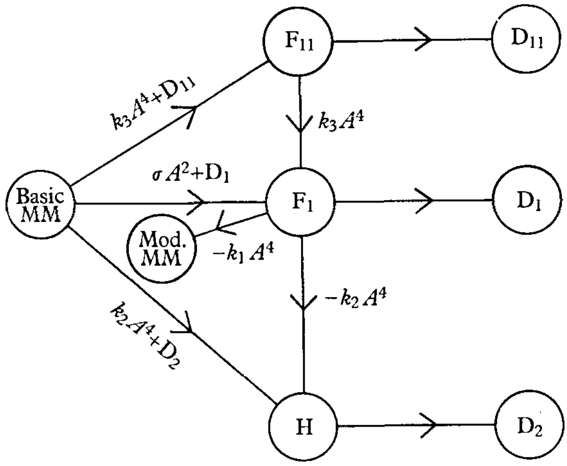

FIGURE 1. Energy supply to and from the fundamental and harmonic to order A^4. (MM = mean motion, F = fundamental, H = harmonic, D = dissipation.)

We compare our theory with the experimental results given in Table 2 of Donnelly & Simon (1960). The torque on the inner cylinder (which is the same as that on the outer cylinder when the motion is steady) may be written

$$G = \frac{2\pi\Omega_1 r_1^3 h\mu}{d}\left|\frac{\partial\bar{v}}{\partial r} - \frac{\bar{v}}{r}\right|_{r=r_1/d}, \tag{5.36}$$

where
$$\left|\frac{\partial\bar{v}}{\partial r} - \frac{\bar{v}}{r}\right|_{r=r_1/d} = \frac{2r_2^2}{r_1(r_1+r_2)} + \delta\left(\frac{r_1+r_2}{2r_1}\right)\left(1-\frac{T_c}{T}\right), \tag{5.37}$$

h is the length of the cylinder and δ is a constant given, ignoring terms of $O(\sigma)$, by

$$\delta = -T_c F_1'(1)/2a_1\epsilon = 0{\cdot}8281. \tag{5.38}$$

We may now rewrite (5.36) and (5.37) in the form

$$G = a\Omega_1^{-1} + b\Omega_1 \quad (\Omega_1 > \Omega_c), \tag{5.39}$$

where
$$\left.\begin{aligned} a &= -9\pi(r_1+r_2)\,h\rho\nu^3\delta T_c/64d^3, \\ b &= \frac{2\pi r_1^3 h\rho\nu}{d}\left\{\frac{2r_2^2}{r_1(r_1+r_2)} + \delta\left(\frac{r_1+r_2}{2r_1}\right)\right\}. \end{aligned}\right\} \tag{5.40}$$

Donnelly used cylinders 5 cm long with radii of 1·0 and 2·0 cm and fluid with $\rho = 0{\cdot}8404\,\mathrm{g\,cm^{-3}}$ and $\nu = 0{\cdot}1226\,\mathrm{cm^2\,sec^{-1}}$. Thus (5.38) and (5.40) give

$$a = -280{\cdot}8 \quad \text{and} \quad b = 12{\cdot}65. \tag{5.41}$$

The curve $G = -280{\cdot}8\Omega_1^{-1} + 12{\cdot}65\Omega_1$ is drawn in figure 2, together with experimental results taken from Table 2 on p. 406 of Donnelly & Simon (1960). For comparison is shown a dashed line given by $G = -333{\cdot}7\Omega_1^{-1} + 13{\cdot}52\Omega_1$ which is obtained by using Stuart's (1958) type of energy balance method. Both these curves give very good agreement with the experimental results for a far wider range of the Taylor number than that over which one would expect the perturbation theory to be useful.

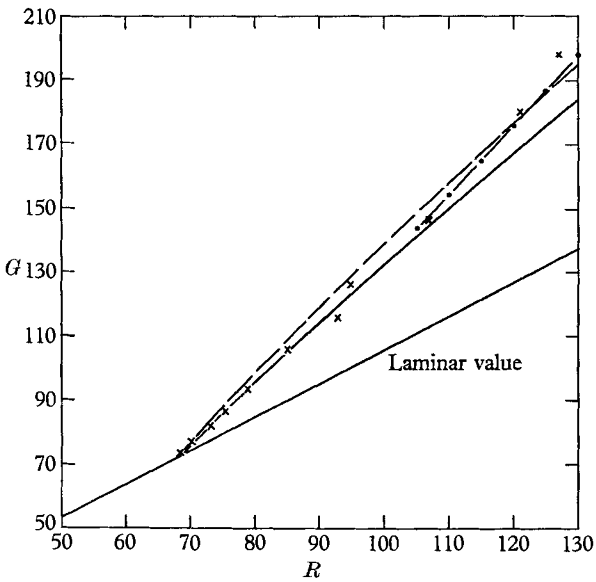

FIGURE 2. Comparison of theories for the wide-gap problem, $r_2 = 2r_1$, $m = 0$, with the experimental results of Donnelly (1958). G, torque *vs* R, Reynolds number. ×, Experiment, Donnelly & Simon (1960); —, present theory; – – –, Stuart's approximate method; —·—, Batchelor's law.

An additional curve of dots and dashes given by $G = \frac{2}{15}R^{1{\cdot}5}$ is included for the higher Reynolds numbers. This curve satisfies the rule $G \sim R^{1{\cdot}5}$, given by Batchelor in an appendix to Donnelly & Simon's paper, taking a base point at $R = 110, G = 154$. Although this range of R is probably not quite high enough for Batchelor's theory to apply, it fares well with the experimental data and matches nicely with the present perturbation theory.

6. The small-gap problem

Here we consider the simplified problem when the gap between the cylinders $d/r_1 \to 0$. Again we determine the shape and strength of the vortices when the amplification rate σ is small (probably compared with λ^2), so that the Taylor number is slightly above its critical value. Again the analysis gives close agreement with experimental results over a wider range of the Taylor number than we expect.

We transform the independent variable measuring distance in the radial direction by setting

$$x = r - \tfrac{1}{2}d^{-1}(r_1 + r_2), \tag{6.1}$$

so that $x = -\frac{1}{2}$ and $\frac{1}{2}$ respectively at the inner and outer cylinders. Using this transformation in (4.1) with (2.8) and taking the limit as $d/r_1 \to 0$, we have for the linear stability problem

$$[(D^2-\lambda^2)(D^2-\lambda^2-\sigma)^2+\lambda^2 T\{1-2x(1-m)/(1+m)\}]\,v_1 = 0, \tag{6.2}$$

with the boundary conditions from (4.2) as

$$v_1 = D^2 v_1 = D(D^2-\lambda^2-\sigma)\,v_1 = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}, \tag{6.3}$$

where now $D \equiv d/dx$. The Taylor number is defined by

$$T \equiv (1-m^2)\,\Omega_1^2\, r_1\, d^3/\nu^2, \tag{6.4}$$

and it is important to note that, since d/r_1 is small, the Reynolds number will be large; also $A_0 = -\frac{1}{2}(1-m)$ from (2.8) with $d/r_1 \to 0$.

As in § 5 it is sufficient to determine the limiting forms as $\sigma \to 0$ of u_1, v_1 and u_2, v_2, and of the adjoint function θ. Also we again fix the wave-number λ at the value λ_c which makes T a minimum when $\sigma = 0$. The determination of λ, T_c is easily accomplished by using either the method of Di Prima (1961) (see, for example, § 7.1) or by using a computer integrating routine (see, for example, § 7.2). However in all cases we use an integrating routine to determine the eigenfunction v_1 and its derivatives in the limit as $\sigma \to 0$. This is done by straightforward integration and one merely ensures that with $v_1'(-\frac{1}{2}) = 1$, say, then $v_1^{(\mathrm{iv})}(-\frac{1}{2})$, $v_1^{(\mathrm{v})}(-\frac{1}{2})$ are chosen to make $v_1(\frac{1}{2}) = v_1''(\frac{1}{2}) = 0$ where now a dash denotes differentiation with respect to x. We then normalize to make $v_1(0) = 1$, and the third boundary condition which should be automatically satisfied gives a check on the accuracy of the eigenvalues and the integration routine.

For convenience we again define $\overline{u}_1$, F_1, $\overline{v}_2$ and $\overline{u}_2$ by (5.5), though now $A_0 = -\frac{1}{2}(1-m)$, and determine the limiting forms of these functions as $d/r_1 \to 0$ and as $\sigma \to 0$. To find $\overline{u}_1$ take these limits in (4.3) and use (5.5) to obtain

$$\overline{u}_1 = (D^2-\lambda^2)\,v_1, \tag{6.5}$$

which, with successive differentials, yields the values of $\overline{u}_1$ and its derivatives since v_1 and its derivatives are known.

Next we find the adjoint function θ by taking the limits $d/r_1 \to 0$ and $\sigma \to 0$ in (4.23) and using $\overline{L}\theta = 0$ so that

$$[(D^2-\lambda^2)^3+\lambda^2 T_c\{1-2x(1-m)/(1+m)\}]\,\theta = 0. \tag{6.6}$$

This in fact is the same equation as is satisfied by v_1 ($\sigma \to 0$); the boundary conditions, found from (4.25), are

$$\theta = D\theta = (D^2-\lambda^2)^2\theta = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{6.7}$$

In § 7.1 where the special case $m \to 1$ is considered then $\theta \equiv \overline{u}_1$, but for other values of m this is not so, although (6.7) are the boundary conditions on $\overline{u}_1$ for any value of m. We fix the magnitude of θ by choosing $\theta''(-\frac{1}{2}) = 1$. Then θ and its derivatives are found by integration from $x = -\frac{1}{2}$ to $x = \frac{1}{2}$. The quantities $\theta'''(-\frac{1}{2})$, $\theta^{(\mathrm{v})}(-\frac{1}{2})$ are chosen so that $\theta(\frac{1}{2}) = \theta'(\frac{1}{2}) = 0$, the third boundary condition again being automatically satisfied.

Now we may determine the relationship between σ and T in the limit as $\sigma \to 0$. When σ is small $v \equiv v_1(x;\, \sigma)$ satisfies (6.2) with the boundary conditions (6.3). Since σ is small let $v = v_1 + \sigma\hat{v} + O(\sigma^2)$ and $T = T_c + \sigma\epsilon + O(\sigma^2)$, where now $v_1 \equiv v_1(x;\, 0)$ and ϵ is a constant to be determined. This is done in exactly the same way as in § 5, by linearizing in σ, multiplying the equation for $\hat{v}$ by $\theta(x;\, 0)$ and integrating over the gap between the cylinders. With use of (5.11) with D^* and D^- both replaced by D we find that

$$\frac{1}{\epsilon} = \frac{\lambda^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \theta\left\{1 - 2x\frac{(1-m)}{(1+m)}\right\} v_1\, dx}{\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} \theta(D^2-\lambda^2)^2 v_1\, dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} v_1(D^2-\lambda^2)^2\,\theta\, dx\right]}. \tag{6.8}$$

This method gave the same result as that of the special case $m \to 1$ in § 7.1 obtained by the method of Di Prima with a discrepancy of less than 1 part in 10^5.

Next we find F_1, which measures the distortion of the mean motion by the Reynolds stress. In (3.15) take the two limits and use (5.5) to obtain

$$D^2 F_1 = D(\bar{u}_1 v_1), \tag{6.9}$$

with the boundary conditions $F_1 = 0$ at $x = \pm\frac{1}{2}$. Thus

$$F_1'(-\tfrac{1}{2}) = -\int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{u}_1 v_1\, dx$$

and, using (6.5) for ease of computation, F_1 and its derivative are found by integration from $x = -\frac{1}{2}$.

Now we determine $\bar{v}_2$ and its derivatives when $d/r_1 \to 0$ and $\sigma \to 0$. Use (5.5) with (4.4), take these limits in turn (terms like u_1^2 can be ignored compared with v_1^2 because $A_0 R$ is very large) to obtain

$$\begin{aligned}&[(D^2-4\lambda^2)^3 + 4\lambda^2 T_c\{1-2x(1-m)/(1+m)\}]\,\bar{v}_2 \\ &\quad = (D^2-4\lambda^2)^2(\bar{u}_1 v_1' - \bar{u}_1' v_1) + 2(\bar{u}_1\bar{u}_1''' - \bar{u}_1'\bar{u}_1'') + 4\lambda^2 T_c\{(1-m)/(1+m)\}\, v_1^2.\end{aligned} \tag{6.10}$$

For ease of computation the right-hand side of (6.10) may be rewritten using (6.5) in terms of v_1. The boundary conditions found from (4.5) are

$$\bar{v}_2 = D^2\bar{v}_2 = D(D^2-4\lambda^2)\,\bar{v}_2 = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{6.11}$$

Now $\bar{v}_2$ and its derivatives are found by integration from $x = -\frac{1}{2}$ to $x = \frac{1}{2}$. A programme was written which, given arbitrary initial values of $\bar{v}_2'(-\frac{1}{2})$, $\bar{v}_2^{(\mathrm{iv})}(-\frac{1}{2})$, $\bar{v}_2^{(\mathrm{v})}(-\frac{1}{2})$, converged to a set of these values which made $\bar{v}_2(\frac{1}{2})$, $\bar{v}_2''(\frac{1}{2})$ and $\bar{v}_2'''(\frac{1}{2}) - 4\lambda^2\bar{v}_2'(\frac{1}{2})$ all zero. The last function we require is the limit as $d/r_1 \to 0$ and $\sigma \to 0$ of $\bar{u}_2$ and this is given by taking these limits in (4.6) and using (5.5) to obtain

$$\bar{u}_2 = (D^2-4\lambda^2)\,\bar{v}_2 + (v_1\bar{u}_1' - v_1'\bar{u}_1). \tag{6.12}$$

From the knowledge of $\bar{v}_2$, $\bar{u}_1$, v_1 and their derivatives and from successive differentials of (6.12) we may also find the derivatives of $\bar{u}_2$.

We may now find the limiting value as $\sigma \to 0$ of the constant a_1. For simplicity and instructiveness we will not use the theory of § 4 but will derive the result from the work of § 3. First define functions $\bar{v}_{11}$, $\bar{u}_{11}$, $\bar{g}_{11}$, $\bar{h}_{11}$ and $\bar{a}_1$ by (5.18) and (5.19).

Then take the limits as $d/r_1 \to 0$ and $\sigma \to 0$ of (3.11, $n = 1$), (3.12, $n = 1$) and use (5.5), (5.18), (5.19) to obtain

$$(D^2-\lambda^2)^2\bar{u}_{11}+\lambda^2T_c\{1-2x(1-m)/(1+m)\}\bar{v}_{11}$$
$$= \bar{a}_1(D^2-\lambda^2)\bar{u}_1+2\lambda^2T_c\{(1-m)/(1+m)\}v_1F_1+\lambda\bar{g}_{11}, \quad (6.13)$$

and
$$(D^2-\lambda^2)\bar{v}_{11}-\bar{u}_{11} = \bar{a}_1v_1+\bar{u}_1F_1'+\bar{h}_{11}, \quad (6.14)$$

where now $\bar{g}_{11}, \bar{h}_{11}$, obtained from (3.13) and (3.14), and using (5.5) and (5.18) are given by

$$4\lambda\bar{g}_{11} \equiv [\bar{u}_1\bar{u}_2'''+2\bar{u}_1'\bar{u}_2''-\bar{u}_1''\bar{u}_2'-2\bar{u}_1'''\bar{u}_2]$$
$$-3\lambda^2[\bar{u}_1\bar{u}_2'+2\bar{u}_1'\bar{u}_2]+4\lambda^2T_c\{(1-m)/(1+m)\}v_1\bar{v}_2, \quad (6.15)$$

and
$$\bar{h}_{11} \equiv [\bar{u}_1'\bar{v}_2+\tfrac{1}{2}\bar{u}_1\bar{v}_2'+\tfrac{1}{2}v_1'\bar{u}_2+\tfrac{1}{4}v_1\bar{u}_2']. \quad (6.16)$$

The boundary conditions on $\bar{u}_{11}, \bar{v}_{11}$ are

$$\bar{u}_{11} = D\bar{u}_{11} = \bar{v}_{11} = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}; \quad (6.17)$$

the full conditions on each function may be found from (6.13), (6.14). The main reason for using (5.19) is, as we shall find in § 7, that a_1 depends mainly upon m through the factor A_0^2 in (5.19) and is relatively little affected by m in the equations which determine $\bar{a}_1$.

Now operate with $(D^2-\lambda^2)^2$ on (6.14) and eliminate $\bar{u}_{11}$ by using (6.13). Multiply the resulting equation by θ and integrate from $x = -\frac{1}{2}$ to $x = \frac{1}{2}$. Then, to simplify, use (6.3) with $\sigma = 0$ and (6.9) and perform some integrations by parts to obtain

$$-\bar{a}_1\left[\int_{-\frac{1}{2}}^{\frac{1}{2}}\theta(D^2-\lambda^2)^2v_1\,dx+\int_{-\frac{1}{2}}^{\frac{1}{2}}v_1(D^2-\lambda^2)^2\theta\,dx\right]$$
$$=\left[\int_{-\frac{1}{2}}^{\frac{1}{2}}\bar{u}_1^2v_1(D^2-\lambda^2)^2\theta\,dx-\left(\int_{-\frac{1}{2}}^{\frac{1}{2}}\bar{u}_1v_1\,dx\right)\left(\int_{-\frac{1}{2}}^{\frac{1}{2}}\bar{u}_1(D^2-\lambda^2)^2\theta\,dx\right)\right]$$
$$+\frac{2(1-m)}{(1+m)}\lambda^2T_c\int_{-\frac{1}{2}}^{\frac{1}{2}}\theta v_1F_1\,dx+\int_{-\frac{1}{2}}^{\frac{1}{2}}\{\lambda\bar{g}_{11}\theta+\bar{h}_{11}(D^2-\lambda^2)^2\theta\}\,dx. \quad (6.18)$$

For a given value of m we may evaluate the integrals in (6.18) and thus determine $\bar{a}_1$. Thus, to first order in σ, we know how the vortices grow together with their equilibrium amplitude, since $A_e^2 = -\sigma/a_1$. Thus, using (6.8) and (5.19), we have

$$A_e^2 = -(8A_0^2T_c/\epsilon\bar{a}_1)(1-T_c/T). \quad (6.19)$$

Numerical results suggest that, at least for $m \geqslant 0$, the variation of $\bar{a}_1$ with m is small; moreover, since the variation of T_c and ϵ is also small, it is clear that A_e is almost proportional to $(1-m)$ for $0 \leqslant m < 1$. (See § 8 for an approximate formula for A_e^2 when $0 \leqslant m < 1$.) Another noteworthy point is that the contribution to $\bar{a}_1$ due to the harmonic terms represented by the last integral on the right-hand side of (6.18) is unlikely to be more than 2% for $0 \leqslant m < 1$. Thus without determining the harmonic functions one can still obtain the value of $\bar{a}_1$ with an error of less than 2 %.

Having determined a_1 we may again show, in a manner identical to that described in § 5, that the differential equation which governs the disturbance amplitude is in fact an energy-balance relation for the fundamental disturbance

(u_1, v_1, w_1). We again denote the 'odd' part of the disturbance by (u', v', w') and the 'even' part by (u'', v'', w''). Both u', w' are of order $(A_0 R)^{-1} v'$ and for a fixed value of $m \neq 1$ it is readily shown using (6.3) that $A_0 R$ may be made arbitrarily large by making d/r_1 sufficiently small. We suppose this condition to be satisfied so that it is sufficient to consider only the energy associated with v', the azimuthal component of the disturbance. Hence we multiply (2.22) by v' and integrate over the space between the cylinders and over one wavelength $(2\pi/\lambda)$ to obtain

$$\frac{1}{R}\frac{\partial}{\partial t}\iint \tfrac{1}{2}(v'^2)\,dr\,dz = \iint(-u'v')\frac{\partial \bar{v}}{\partial r}\,dr\,dz - \frac{1}{R}\iint(\xi'^2+\zeta'^2)\,dr\,dz - \iint v'\chi_{21}\,dr\,dz, \quad (6.20)$$

where ξ', ζ' are given by (5.26) and χ_{21} by (5.28). In obtaining (6.20) the factor r has been removed throughout and $\bar{v}/r$ has been ignored compared with $\partial\bar{v}/\partial r$. In a similar manner to that described in § 5 we may now, using x as the independent variable, show that

$$k_1 = -\frac{1}{8A_0^2 k_0}\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{u}_1^2 v_1^2\,dx - \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{u}_1 v_1\,dx\right)^2\right], \quad (6.21)$$

$$k_2 = -\frac{1}{8A_0^2 k_0}\int_{-\frac{1}{2}}^{\frac{1}{2}} v_1(\bar{u}_1'\bar{v}_2 + \tfrac{1}{2}\bar{u}_1\bar{v}_2')\,dx, \quad (6.22)$$

$$k_3 = -\frac{1}{8A_0^2 k_0}\int_{-\frac{1}{2}}^{\frac{1}{2}} (v_1\bar{u}_{11} - \bar{u}_1 v_{11})\,dx, \quad (6.23)$$

where
$$k_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} v_1^2\,dx. \quad (6.24)$$

A consistency check that $a_1 = k_1 + k_2 + k_3$ is readily obtained by using (6.14). This is done by multiplying (6.14) throughout by v_1 and integrating from $x = -\frac{1}{2}$ to $x = \frac{1}{2}$.

Now $8A_0^2 k_1$, $8A_0^2 k_2$ may be evaluated directly from (6.21), (6.22) and, since $a_1 = k_1 + k_2 + k_3$, we may use (5.19), (6.18) to determine $8A_0^2 k_3$. The signs and relative magnitudes of k_1, k_2 and k_3 in this special small-gap problem differ markedly from the wide-gap problem. The numerical results of §§ 7.1, 7.2 indicate that, for all values of m, k_1 is negative, again representing flow of energy to the disturbance from the mean motion. In the limit as $m \to 1$, it is found that k_2/k_1 is approximately 0·0073 and, when $m = 0$, k_2/k_1 is approximately $-0{\cdot}0003$; hence k_2 may be positive or negative but it seems that k_2/k_1 is always small so that this effect is relatively unimportant. Moreover, in the limit as $m \to 1$, k_3/k_1 is approximately $-0{\cdot}34$; and, when $m = 0$, k_3/k_1 is approximately $-0{\cdot}38$. Thus it seems that k_3 is probably positive for all values of m and the distortion of the fundamental is important and tends to increase the equilibrium amplitude. Using (6.18), (6.21) and (6.22) one can also show that the contribution (k_{32}) to k_3 from the harmonic is very small, so that, as a whole, the contribution to a_1 due to the harmonic is small (less than 2 % for all values of m).

Although the harmonic only plays a minor role it does play an interesting one. The work of § 7.1 indicates that for values of m near 1 the harmonic derives all its energy from the fundamental as we might expect intuitively. About 75 % of this is lost by viscous dissipation and the rest is transferred by the harmonic back into the mean motion. However, the work of § 7.2 indicates that for values of m near 0, the harmonic derives its energy from the mean motion, about 0·3 % of this energy is transferred to the fundamental and the rest is lost by viscous dissipation. Thus in these cases, although the harmonic cannot exist without the influence of the fundamental, the latter plays the role of a 'catalyst'.

7.1. Results for the small-gap problem when $m \to 1$

Here we give the numerical results of the small-gap problem discussed in § 6 when $m \to 1$ from below. This means that the cylinders rotate with nearly the same angular speeds.

Setting $m = 1$ in (6.2), the equation for v_1 is

$$[(D^2-\lambda^2)(D^2-\lambda^2-\sigma)^2+\lambda^2 T]\,v_1 = 0, \tag{7.1.1}$$

the boundary conditions being (6.3). As a consequence of the constancy of the coefficients of (7.1.1) v_1 may be taken to be an even function, since it is known that instability corresponding to the odd modes only occurs at much higher Taylor numbers than those considered here. The eigenvalues λ, T_c and the relationship between σ and $T-T_c$ were obtained by the method of Di Prima which gave the following transcendental equation for T as a function of λ^2 and σ, that is

$$\sum_{n=1}^{\infty}\frac{(2n-1)^2(A_n+\sigma)}{A_n(A_n+\sigma)^2-\lambda^2 T} = 0, \quad \text{where} \quad A_n \equiv (2n-1)^2\pi^2+\lambda^2. \tag{7.1.2}$$

Thus the neutral curve, obtained by setting $\sigma = 0$ in (7.1.2) is given by

$$\sum_{n=1}^{\infty}\frac{(2n-1)^2 A_n}{A_n^3-\lambda^2 T} = 0, \tag{7.1.3}$$

which series converges like $(2n-1)^{-2}$. However, with an error of less than 1 part in 10^8, we write (7.1.3) in the form

$$1+8\pi^2\sum_{n=1}^{20}\left\{\frac{(2n-1)^2 A_n}{A_n^3-\lambda^2 T}-\frac{1}{\pi^4(2n-1)^2}\right\}-\frac{\lambda^2}{24\,000\pi^4} = 0; \tag{7.1.4}$$

and using the computer this gave a minimum value of $T = T_c = 1707{\cdot}77$ when $\lambda = 3{\cdot}12$, in good agreement with the value obtained by Pellew & Southwell (1940). Thus we fix λ to be 3·12 supposing that the basic disturbance has this wave-number.

From (7.1.2) it may be shown that, with an error of less than 1 part in 10^5, the relation in the limit as $\sigma \to 0$ between σ and T is $\sigma G = (1-T_c/T)H$ where G, H may be evaluated from

$$G = 1+96\pi^2\sum_{n=1}^{10}\left[\frac{(2n-1)^2(A_n^3+\lambda^2 T_c)}{(A_n^3-\lambda^2 T_c)^2}-\frac{1}{\pi^6(2n-1)^4}\right]-\frac{9\lambda^2}{10^6\pi^6}, \tag{7.1.5}$$

and

$$H = \lambda^2 T_c\sum_{n=1}^{5}\left[\frac{(2n-1)^2 A_n}{(A_n^3-\lambda^2 T_c)^2}\right], \tag{7.1.6}$$

the values obtained being $G = 3{\cdot}0450\times 10^{-4}$ and $H = 3{\cdot}9613\times 10^{-3}$, so that

$$\sigma = 13{\cdot}01(1-T_c/T). \tag{7.1.7}$$

Since v_1 is an even function then $v_1(0) = 1$, $v_1'(0) = v_1'''(0) = v_1^{(v)}(0) = 0$ and it was only necessary to integrate from $x = 0$ to $x = \frac{1}{2}$ to make $v_1(\frac{1}{2}) = v_1''(\frac{1}{2}) = 0$. From (6.5) $\bar{u}_1$ is clearly also an even function and by operating on (7.1.1) with $(D^2 - \lambda^2)$ it follows that $\bar{u}_1$ satisfies the same equation as v_1. Hence $\bar{u}_1$ was found, together with its derivatives in the same way as v_1 and its derivatives were found, and (6.5) was used as a check on the results. Values of v_1, $\bar{u}_1$ and their first three derivatives are to be found in table 5.

From (6.6) with $m = 1$ and from (6.7) it is clear that $\theta \equiv \bar{u}_1$ so that, for $m = 1$, $\bar{u}_1$ is the adjoint function. Thus setting $\theta = \bar{u}_1$ in (6.8) with $m = 1$ and evaluating the integrals, we obtained a check on (7.1.7).

From (6.9) the function F_1 is clearly odd so that

$$F_1 = \int_0^x \bar{u}_1 v_1 \, dx - 2x \int_0^{\frac{1}{2}} \bar{u}_1 v_1 \, dx; \tag{7.1.8}$$

the function and its derivative are to be found in table 6.

Next we determined $\bar{v}_2$ and its derivatives from (6.10) with $m = 1$, from which it is evident that $\bar{v}_2$ is an odd function. Thus $\bar{v}_2(0) = \bar{v}_2''(0) = \bar{v}_2^{(iv)}(0) = 0$ and we integrated from $x = 0$ to $x = \frac{1}{2}$ choosing $\bar{v}_2'(0)$, $\bar{v}_2'''(0)$, $\bar{v}_2^{(v)}(0)$ so as to make $\bar{v}_2(\frac{1}{2}) = \bar{v}_2''(\frac{1}{2}) = \bar{v}_2'''(\frac{1}{2}) - 4\lambda^2 \bar{v}_2'(\frac{1}{2}) = 0$. In general when $m \neq 1$ the sixth-order differential equation for $\bar{u}_2$ is rather complicated which is why we normally use (6.12). However, when $m = 1$ it is readily shown that $\bar{u}_2$ may be found directly from

$$[(D^2 - 4\lambda^2)^3 + 4\lambda^2 T_c]\,\bar{u}_2 = 6\bar{u}_1 \bar{u}_1^{(v)} - 2\bar{u}_1' \bar{u}_1^{(iv)} - 4\bar{u}_1'' \bar{u}_1''' + \lambda^2(-16\bar{u}_1 \bar{u}_1''' + 16\bar{u}_1' \bar{u}_1''). \tag{7.1.9}$$

Thus $\bar{u}_2$ is also odd and we found $\bar{u}_2$ by integrating from $x = 0$ to $x = \frac{1}{2}$ and choosing $\bar{u}_2'(0)$, $\bar{u}_2'''(0)$, $\bar{u}_2^{(v)}(0)$ so as to make $\bar{u}_2(\frac{1}{2}) = \bar{u}_2'(\frac{1}{2}) = \bar{u}_2^{(iv)}(\frac{1}{2}) - 8\lambda^2 \bar{u}_2''(\frac{1}{2}) = 0$. The functions $\bar{v}_2$, $\bar{u}_2$ and their first three derivatives are to be found in table 7. These results were found to check with (6.12) on using values given in table 5.

We now find $\bar{a}_1$ from (6.18) with $\theta = \bar{u}_1$ and with $m = 1$ in (6.15) and (6.18). We find that

$$\bar{a}_1 = -85{\cdot}39, \tag{7.1.10}$$

where the contribution to $\bar{a}_1$ due to the harmonic, and represented by the last integral on the right-hand side of (6.18), is $-1{\cdot}33$. Thus, as mentioned in § 6, we see that the contribution to $\bar{a}_1$ due to the harmonic is less than 2 %. Hence, using $A_e^2 = -\sigma/a_1$, with (5.19) and (7.1.7), it follows that for values of m near 1

$$A_e^2 = 0{\cdot}3047(1-m)^2(1 - T_c/T) \quad (m \to 1), \tag{7.1.11}$$

with the normalizing condition $v_1(0) = 1$.

We may also determine the contributions to $\bar{a}_1$ from k_1, k_2 and k_3 as mentioned in § 6 and we find that

$$8A_0^2 k_1 = -127{\cdot}11, \quad 8A_0^2 k_2 = -0{\cdot}92, \quad 8A_0^2 k_3 = 42{\cdot}64. \tag{7.1.12}$$

These results confirm the remarks of § 6 that, for $m \to 1$, k_1 is negative, that k_2/k_1 is small and that $k_3/k_1 = -0{\cdot}336$ is quite large and negative. By considering an equation similar to (6.20) for the 'even' part of the disturbance and evaluating some integrals we may show that about 75 % of the energy supplied to the harmonic by the fundamental is lost in viscous dissipation and the rest is fed into the mean motion.

7.2. Results for the small-gap problem when $m = 0$

Here we give the detailed numerical results of the small-gap problem when $m = 0$. Setting $m = 0$ and $\sigma = 0$ in (6.20) the equation for v_1 is

$$[(D^2-\lambda^2)^3+\lambda^2 T_c(1-2x)]\,v_1 = 0, \tag{7.2.1}$$

so that the operator no longer has constant coefficients. The boundary conditions from (6.3) with $\sigma = 0$ are

$$v_1 = D^2 v_1 = D(D^2-\lambda^2)\,v_1 = 0 \quad \text{at} \quad x = \pm\tfrac{1}{2}. \tag{7.2.2}$$

Unlike the case $m \to 1$, v_1 is not 'even'; but we shall see, however, that the 'odd' part is relatively small.

To determine the eigenvalues a programme was written which, given a value of λ together with arbitrary values of T, $v_1^{(\text{iv})}(\frac{1}{2})$ and $v_1^{(\text{v})}(-\frac{1}{2})$ (and with $v_1'(-\frac{1}{2}) = 1$ specified), converged to a set of these three quantities which made

$$v_1(\tfrac{1}{2}) = v_1''(\tfrac{1}{2}) = v_1'''(\tfrac{1}{2})-\lambda^2 v_1'(\tfrac{1}{2}) = 0$$

when (7.2.1) was integrated from $x = -\frac{1}{2}$ to $x = \frac{1}{2}$. This gave a plot of the neutral curve and the programme also selected the value of λ which made T a minimum. The values obtained were $\lambda = 3{\cdot}13$ and $T = T_c = 1694{\cdot}95$. At the same time v_1 and its derivatives were also found, these were then normalized so that $v_1(0) = 1$. Then $\bar{u}_1$ and its derivatives were found from (6.5) and its differentials. The values of v_1, $\bar{u}_1$ and their first three derivatives are to be found in table 8.

Next we found the adjoint function θ from (6.6) with $m = 0$ and from the boundary conditions (6.7). This was done as explained in § 6, and θ and its first three derivatives are to be found in table 9. The relationship between σ and T in the limit as $\sigma \to 0$ was then found from (6.8) with $m = 0$, and evaluating the integrals we obtained

$$\sigma = 13{\cdot}10(1-T_c/T). \tag{7.2.3}$$

The function F_1 and its derivative were then found as indicated in § 6 and these are to be found in table 10.

Hence we then found $\bar{v}_2$ and its derivatives from (6.10) with $m = 0$, and $\bar{u}_2$ together with its derivatives from (6.12) as indicated in § 6. These functions and their first three derivatives are to be found in table 11.

Then $\bar{a}_1$ was found from (6.18) with $\bar{g}_{11}$ given by (6.15) with $m = 0$ and $\bar{h}_{11}$ given by (6.16). Evaluating the integrals in (6.18) we obtained

$$\bar{a}_1 = -80{\cdot}44, \tag{7.2.4}$$

where the contribution ($\bar{a}_{12}$) to $\bar{a}_1$ due to the harmonic terms represented by the last integral on the right-hand side of (6.18) was $-1{\cdot}03$. Thus, as in the case $m \to 1$, the contribution to $\bar{a}_1$ due to the harmonic when $m = 0$ is again less than 2 %. Now using $A_e^2 = -\sigma/a_1$, together with (5.19) and (7.2.3), it follows that

$$A_e^2 = 0{\cdot}3257(1-T_c/T) \quad (m = 0) \tag{7.2.5}$$

with the normalizing condition $v_1(0) = 1$. The contributions of k_1, k_2, k_3 to $\bar{a}_1$ are now found as indicated in § 6 and

$$8A_0^2 k_1 = -129{\cdot}33, \quad 8A_0^2 k_2 = 0{\cdot}04, \quad 8A_0^2 k_3 = 48{\cdot}85. \tag{7.2.6}$$

These results are similar to those of the case $m \to 1$ in that k_1 is negative, k_2/k_1 is very small and k_3/k_1 is quite large and positive so that as before the distortion of the fundamental tends to increase the equilibrium amplitude. Unlike the case $m \to 1$, however, k_2 is now positive which means that energy flows from the harmonic to the fundamental. By considering an energy equation similar to (6.20) for the 'even' part of the disturbance we may show that the harmonic extracts energy from the mean motion; over 99 % of this energy is lost in viscous dissipation and the rest is that which is transferred to the fundamental.

x	v_1	$10^{-1}\bar{u}_1$	$10^2\theta$	F_1	$\bar{v}_2$	$10^{-1}\bar{u}_2$
−0·50	0·000	0·000	0·000	0·000	0·000	0·000
−0·45	0·144	−0·093	0·112	0·468	−0·138	0·144
−0·40	0·288	−0·326	0·397	0·914	−0·279	0·514
−0·35	0·431	−0·643	0·789	1·295	−0·417	1·035
−0·30	0·569	−0·995	1·232	1·557	−0·542	1·641
−0·25	0·695	−1·340	1·677	1·654	−0·644	2·271
−0·20	0·806	−1·647	2·085	1·559	−0·718	2·869
−0·15	0·895	−1·894	2·426	1·270	−0·761	3·387
−0·10	0·958	−2·063	2·678	0·817	−0·773	3·789
−0·05	0·994	−2·147	2·825	0·253	−0·756	4·048
0·00	1·000	−2·144	2·859	−0·353	−0·714	4·151
0·05	0·977	−2·055	2·781	−0·928	−0·653	4·095
0·10	0·927	−1·892	2·596	−1·404	−0·579	3·887
0·15	0·853	−1·665	2·318	−1·730	−0·497	3·542
0·20	0·758	−1·390	1·964	−1·880	−0·411	3·078
0·25	0·646	−1·088	1·559	−1·848	−0·327	2·521
0·30	0·524	−0·779	1·131	−1·654	−0·248	1·901
0·35	0·394	−0·488	0·717	−1·331	−0·176	1·262
0·40	0·262	−0·240	0·357	−0·922	−0·112	0·663
0·45	0·130	−0·066	0·100	−0·468	−0·054	0·197
0·50	0·000	0·000	0·000	0·000	0·000	0·000

TABLE B. Summary of results for the small-gap problem, $m = 0$, with $v_1(0) = 1$ and $\theta''(-0{\cdot}5) = 1$.

Comparison with experiment for $m = 0$

Now that the amplitude of the velocity distribution is known, together with the distortion of the mean motion, we may calculate the torque required to maintain the motion.

Taylor (1936) carried out experiments with cylinders 84·4 cm long, the cylinders having radii of 3·94 and 4·05 cm. In figure 3, G denotes the torque measured in g cm^2 sec^{-2} units, and N the angular speed measured in units of rev. sec^{-1}. The broken line corresponds to the theory of Stuart (1958) but modified to include the terms of $O(d/r_1)$ in the 'laminar' torque. Both curves give good agreement with the experimental results for a fairly wide range of the Taylor number above the critical value.

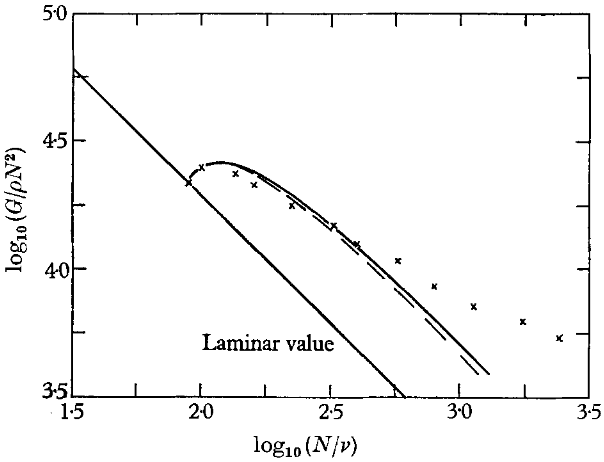

FIGURE 3. Comparison of theories for the small-gap problem, $m = 0$, with the experimental results of Taylor (1936). ×, Experiment, Taylor (1936); —, present theory; ---, Stuart's approximate method.

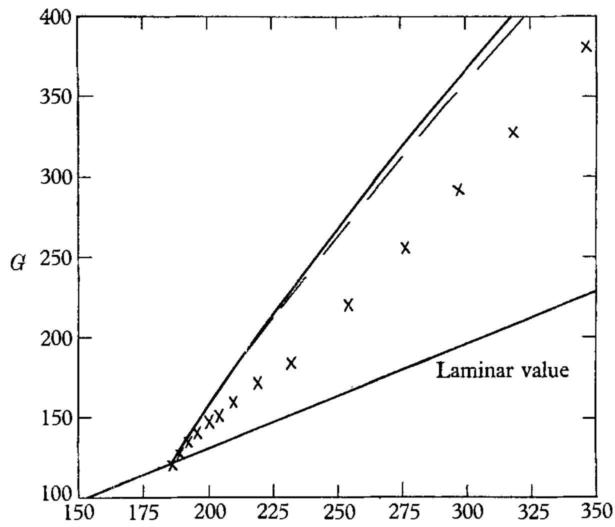

FIGURE 4. Comparison of theories for the small-gap problem, $m = 0$, with the experimental results of Donnelly (1958). G, torque *vs* R, Reynolds number. ×, Experiment, Donnelly & Simon (1960); —, present theory; ---, Stuart's approximate method.

Finally we compare our theory with the experimental results given in Table 1 of Donnelly & Simon (1960). The torque on the inner cylinder is given by (5.36) and (5.37) where now

$$\delta = -T_c F_1'(-\tfrac{1}{2})/2a_1\epsilon = 1{\cdot}528. \qquad (7.2.7)$$

Donnelly used cylinders 5 cm long with radii of 1·9 and 2·0 cm and fluid with $\rho = 1{\cdot}585\,\mathrm{g\,cm^{-3}}$ and $\nu = 5{\cdot}796 \times 10^{-3}\,\mathrm{cm^2\,sec^{-1}}$. Rewriting (5.36) and (5.37) in the form (5.39) and ignoring terms in the coefficient of δ which are $O(d/r_1)$, we obtained

$$G = -906\cdot6\Omega_1^{-1} + 51\cdot64\Omega_1. \qquad (7.2.8)$$

Before comparing with experiment we adjusted (7.2.8) so that, using (7.11) of Taylor (1923), terms of $O(d/r_1)$ were included in the determination of T_c. Such terms were not however incorporated in the non-linear effects so that δ was not adjusted.

The bold line in figure 4 compares the theory with Donnelly's experimental results, and the broken line represents Stuart's theory ($\delta = 1\cdot447$), modified to include terms of $O(d/r_1)$ in the 'laminar' torque and also in T_c. Both curves give good agreement with experiment over the range of the Taylor number above the critical value for which we expect our perturbation theory to be valid.

8. Discussion of the results

The results of the wide-gap problem of § 5 with $r_2 = 2r_1$ and $m = 0$ clearly indicate that the theory presented in this paper gives very close agreement with experimental results over the range of σ, small compared with λ^2, for which we expect the theory to be valid. We suggest that over the corresponding ranges of σ for different values of r_2/r_1 and m this will also be true. In § 5 we also found that the theory agrees with experiment for much larger values of σ than we expect. Whether this will be true for all wide-gap problems is not certain. It also seems probable that for most wide-gap problems the generation of the harmonic will affect the equilibrium amplitude more than the distortion of the fundamental.

The results of the small-gap problems of §§ 7.1 and 7.2 also indicate that over the range of validity of the perturbation theory accurate agreement is obtained with experiment. That such good agreement is not obtained for larger values of σ is probably due to non-axisymmetric disturbances which, when $r_1/r_2 = 0\cdot95$, have been observed experimentally by Donnelly (private communication) at about $R = 1\cdot1R_c$. (Related observations have also been obtained by Coles 1960.) These will presumably contribute to the torque. However in the wide-gap problem of § 5 the experiments of Donnelly & Fultz (1960) indicated that non-axisymmetric disturbances were not noticeable when $R < 5R_c$. We note that when the gap is small the distortion of the fundamental affects the equilibrium amplitude more than the generation of the harmonic, in contrast to the wide-gap problem.

The results of §§ 7.1 and 7.2 and an analysis of the terms in equation (6.18) enables us to propose the following approximate formulae valid for $0 \leqslant m < 1$. Omitting the second term on the right-hand side of (6.18) gives $-\bar{a}_1 = 85\cdot28$ which is almost the same as the value 85·39 given in (7.1.10) when $m = 1$ and this term is zero. Thus we may suppose that the variation of $\bar{a}_1$ with m is mainly due to this term which varies like $(1-m)^2/(1+m)^2$, so that if A, B are constants we set $\bar{a}_1 = A + B(1-m)^2/(1+m)^2$. Fitting this with the results for $m = 1$, $m = 0$ we obtain, with a probable error of less than 1%, that

$$\bar{a}_1 = -85\cdot4 + 5\cdot0(1-m)^2/(1+m)^2 \quad (0 \leqslant m < 1). \qquad (8.1)$$

An examination of (6.8) with use of the results for $m = 1$, $m = 0$ also indicates with about the same error that

$$T_c/\epsilon = 13\cdot0+0\cdot1(1-m)^2/(1+m)^2 \quad (0 \leqslant m < 1). \tag{8.2}$$

Hence using (6.19) with (7.2) and (7.3) we propose that

$$A_e^2 = 0\cdot305[1+0\cdot067(1-m)^2/(1+m)^2](1-T_c/T) \quad (0 \leqslant m < 1). \tag{8.3}$$

The value of T_c to use in (8.3) may be obtained from the Taylor formula, written in a slightly different form, namely

$$T_c = 1708-13(1-m)^2/(1+m)^2. \tag{8.4}$$

For corrections to (8.4) to account for terms of $O(d/r_1)$ use may be made of equation (7.11) of Taylor (1923).

By using, with $m = 1$, (6.8) and (6.18) without the last term on the right-hand side (this represents the contribution to $\bar{a}_1$ due to the harmonic, about 1·7 %) we may write

$$A_e^2 = \frac{-8A_0^2\left(1-\frac{T_c}{T}\right)\int_0^{\frac{1}{2}}\bar{u}_1 v_1\,dx}{\left[\int_0^{\frac{1}{2}}\bar{u}_1^2 v_1^2\,dx - 2\left(\int_0^{\frac{1}{2}}\bar{u}_1 v_1\,dx\right)^2\right]}, \tag{8.5}$$

which gives a result for A_e^2/A_0^2 identical with that obtained by Stuart (1958), who ignored the harmonic and the distortion of the fundamental and based his calculations on the neutral curve. Although one might expect Stuart's method to give substantial errors in determining the equilibrium amplitude, it does not do so. His method gives an equation like

$$\tfrac{1}{2}\,dA^2/dt = \sigma' A^2 + k_1 A^4, \tag{8.6}$$

where k_2, k_3 and terms of order σ have been omitted from the right-hand side; σ' is not the correct amplification rate of infinitesimal disturbances. However it turns out (when $m = 1$) that

$$\sigma'/k_1 = \sigma/(k_1+k_{31}), \tag{8.7}$$

where k_{31} is the part (about 98 %) of k_3 excluding that (k_{32}) due to the harmonic. Thus the two main deficiences of Stuart's method cancel each other in the equilibrium state, though separately they are each substantial. This indicates that it is a very good approximation, when calculating the equilibrium amplitude, to use Stuart's method for any value of $m \geqslant 0$. It is clear, from figure 2, that Stuart's method also gives good results in the wide-gap problem of § 5.

The question may be raised as to the necessity or desirability of studying the time-dependent problem, as is done in this paper, when comparison with experiment has been made only in the equilibrium (steady) state; moreover, the latter may be calculated by a method equivalent to that of Malkus & Veronis (1958) for the thermal-convection problem. There are three main reasons for considering the time-dependent problem. First, the additional algebra required is comparatively small, and leads to a very similar numerical problem to that of the steady case. Secondly, a study of the time dependence puts the amplified solutions of linearized theory in perspective with the present finite-amplitude analysis. Thirdly, if such work is to be extended to consider the relative stability of different modes at finite amplitudes then it is vital to study the time dependence; see, for example, the work of Segel & Stuart (1962) on preferred modes in the thermal-convection problem.

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